Embedding Problems

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Definition 1. *(Equivalence of embeddings) Two embeddings $f, g : X \to Y$ are equivalent if there is a homeomophism $h : Y \to Y$ s.t. $h \circ f(x) = g(x)$*

Problem 1. *Give an example of non-equivalent embeddings $X \to Y$ where X is a) connected b) disconnected*

Problem 2. *Let $f : S^1 \to S^1 \times \mathbb{R}$ be an embedding s.t.$f(S^1) = S^1 \times \{0\}$ then if $g : S^1 \to S^1 \times \mathbb{R}$ is equivalent to f it sends the generator of $H_1(S^1)$ to the generator of $H_1(S^1 \times \mathbb{R})$. Thus, describe the equivalence classes of embeddings $S^1 \to S^1 \times \mathbb{R}$. Is theanswer the same for $S^1 \to S^1 \times \mathbb{R}^n$ with $n > 1$?*

Problem 3. *Do equivalent embedding have homeomorphic complements? That is given two embeddings $g, f : X \to Y$ then isit truw that $Y \setminus f(X) \cong Y \setminus g(X)$?*

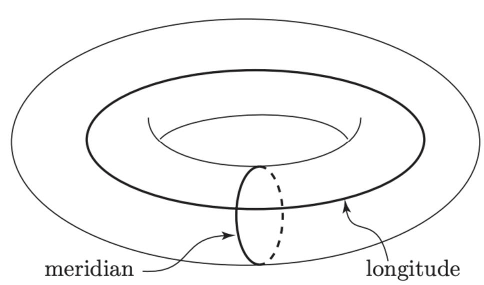
Definition 2. *(Flat) A k sphere/cell S in $\mathbb{R}^n$ is flat if there is some homeomorphism h of $\mathbb{R}^n$ s.t. $h(S) = S_{st}$ where S_{st} is the standard unit sphere or disk in $\mathbb{R}^n$.*

Definition 3. *(Locally Flat) A k sphere/cell S in $\mathbb{R}^n$ is locally flat at $x \in S$ if there is an open neighbourhood U of x and a homeomorphism of pairs $(U, U \cap S) \cong (\mathbb{R}^n, \mathbb{R}^k)$*

Problem 4. *Show that if a k sphere/cell S in $\mathbb{R}^n$ is locally flat at all of it's points then it is flat.*

Definition 4. *(Knotted embeddings) Fix some standard embedding $X \to Y$ then an embedding $f : X \to Y$ is knotted if it is homotopic to the standard embedding but not equivalent to it.*

Problem 5. *(Torus Knots) The standard embedding for a torus $T = S^1 \times S^1$ in $\mathbb{R}^3$ is the one shown in the diagram below. A curve of the form $S^1 \times pt$ is called a longitude and a curve of the form $\{pt\} \times S^1$ is called a meridian. Let $p, q \in \mathbb{Z}$ be relatively prime.*



a) show that there exists a simple closed curve $K_{p,q}$ that is homologous to p times longitude plus q times the meridian. This is called the (p,q) Torus knot. Also show that $T^2 \setminus K_{p,q} \cong S^1 \times (-1,1)$.

b)(*) For which values of p,q is $K_{p,q}$ flat or unkotted?

Definition 5. *(Tame and Wild Cantor Sets) A Cantor set $C \subset \mathbb{R}^n$ is tame if there is a homeomorphism of $\mathbb{R}^n$ taking it to the standard middle thirds Cantor set in some linear subspace. (Note this is the same as flat). Else the Cantor set is wild.*

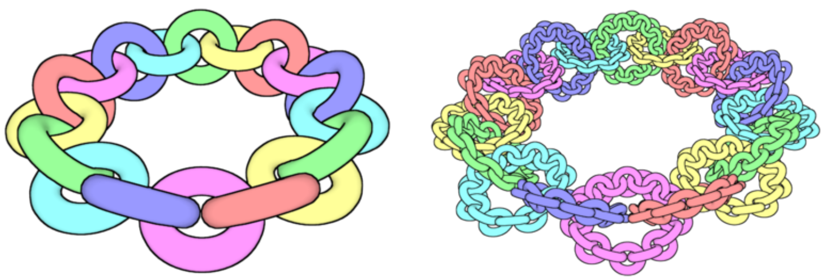
Problem 6. *Let $C \subset \mathbb{R}^n$ be a Cantor set. Show that for:*

a) $n = 1$ there is a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h(C)$ is the middle thirds Cantor set

b) $n = 2$ there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(C) \subset l$ for some affine line l.

So conclude that every Cantor set in $\mathbb{R}^2$ is tame.

Problem 7. *(Antoine Cantor Sets) Let $T_0 \subset \mathbb{R}^3$ be a solid torus and let T_1 be a collection of at least 3 solid tori s.t. each links only its two neighbours. For each solid torus component of T_1 find collections of at least 3 solid tori linked as before and let T_2 be the union of these collections of tori that lie in each component of T_1. Proceed to define $T_3 \supset T_4 \supset \dots$ in a similar manner. Define the Antoine Cantor set as the intersection $T = \cap T_i$. Below are T_1 and T_2.*



a) Show that T is a wild Cantor set and that no 2-sphere separates it.

b) Ammend the construction to obtain a wild arc and a wild sphere in $\mathbb{R}^3$

c) Extend this construction to $\mathbb{R}^n$

d)(*) Show that there are an uncountable number of inequivalently embedded Cantor sets

Problem 8. *Vary Antoine's construction. Now we allow T_i to be any number of tori. For example we could pick T_i to be a single solid torus inside T_{i-1} and homothetic to it. In this case $T = S^1$.*

a) Can you still obtain wild Cantor sets if at each stage in the construction T_i can only have one or two solid tori in each torus component of T_{i-1}

b) Construct a wild Cantor set with a simply connected complement.

c)(*) Give an example of a contractible open 3-manifold $M \subset \mathbb{R}^3$ that is not homeomorphic to $\mathbb{R}^3$

Problem 9. *(Daverman's question)(***) Mazur constructed contractible compact smooth 4-manifolds in $\mathbb{R}^4$ that that have boundary a homology 3-sphere not homeomorphic to the standard S^3. Likorish showed further that it was possible to knot these manifolds in $\mathbb{R}^4$. Combining their results it may now be posssible to answer a question of Daverman: Is there a collection of contractible 4 manifolds $M_i \subset \mathbb{R}^4$ s.t. their intersection gives a wild Cantor set?*

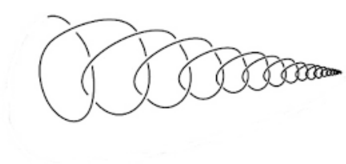
Problem 10. *Given a Cantor set $C \subset \mathbb{R}^n$ show that there exists an arc α s.t. $C \subset \alpha \subset \mathbb{R}^n$.*

Definition 6. *(Pierced by a segment) A sphere $S \in \mathbb{R}^n$ is said to be pierced by a segment if there is a straight line l (according to the affine structure) s.t. $l \cap S = \{pt\}$.*

Problem 11. *A previous question asked at a conference in Stockholm conjectured that a sphere $S \subset \mathbb{R}^3$ pierced by a segment at each of it's points is flat. Prove or disprove the conjecture. (Is the Horned sphere a counterexample?)*

Definition 7. *(Cellular) A subset X of $\mathbb{R}^n$ is cellular if there exists a sequence of nested n-balls $B_n \supset Int(B_{n+1})$ that intersect in X i.e. $X = \cap B_i$*

Problem 12. *(Fox-Artin) Recall Fox-Artin's construction of an arc wild in one point (see below).*
a) Show that it is cellular explicitly
b) it's double is wild




Problem 13. *Consider the infinite trefoil knot drawn in the diagram above. Show that it is not flat. What happens if we wedge it with an interval at the limit point?*

Problem 14. *(**) The spheres of Fox and Artin give examples of wild embeddings of S^2 in S^3 that fail to be flat at a finite number of points. Is it possible to construct examples of n-spheres in S^{n+1} that are wild at a finite number of points ($n > 2$) either by amending the construction above or by other methods.*

Problem 15. *(Brown) A question of Brown in the 60s asked whether if $A \vee B$ is cellular then A is cellular. Using the wild constructions discussed previously or otherwise answer Brown's question.*

Problem 16. *(Rolfsen's Problem) Rolfsen's Problem asks whether any knot is topologically isotopic to the unknot.*
a) Show that a knot that is locally flat at one point is isotopic to the unknot.
b) Consider the pseudo-isotopy that contracts the Bing Sling onto the centre of its first defining torus. Can it be extended to an isotopy?
c) constrcut a knot that pierces no disc but is isotopic to the unknot.