The Haar Measure

Existence and uniqueness of the Haar Measure on Locally Compact Groups

Sigrid Schrøder Jensen,

February 11, 2023

Lecture notes for Farum Seminar February 2023

Contents

Chapter 1

Introduction

The goal of these lecture notes is to show Haar's Theorem, concerning the existence and uniqueness of the Haar measure on locally compact Hausdorff topological groups. It was originally written as a proof of participation in the course Analysis on Topological Groups at Aalborg University in 2022 by Esben Dissing Bregnballe, Laurits Gorm Dahl, Sigrid Schrøder Jensen, Mads Bjerregaard Kjær, and Marcus Johan Schytt, and later corrected by Morten Grud Rasmussen. The text has now been reworked into lecture notes to follow a short course on the Haar Measure, held by Sigrid Schrøder Jensen, at the February 2023 Farum Seminar.

Most proofs in the text are left as exercises for the reader. The few proofs left untouched serve as guides to similar proofs left for the reader. This is with the exception of Sections 5.2 and 5.3, regarding the uniqueness of the Haar measure, which are left fully untouched. These sections require a lot more general integration and functional analysis theory that will not be needed to participate in the course. They will not be covered in the course, but are left as additional literature for the curious after the course has been concluded.

The course will be split up into two pars: One following Chapters 3 and 4, and one following Chapter 5 until the start of Section 5.2. The reader following the first part is assumed to be familiar with basic topology, and groups and their operations. If the reader has not yet been introduced to the notion of topological spaces, Chapter 2 serves as an introduction, covering only the exact concepts needed to understand the following two chapters. This chapter will not be covered in the course. The reader following also the second part is assumed to have finished a first course in measure theory.

Chapter 2

Topological spaces

This chapter serves as a non-comprehensive introduction to topology, giving only the exact tools that one would need to work through Chapters 3 and 4, without any previous knowledge about topology. In this chapter, the reader is assumed to be familiar with basic set theory and analysis. This chapter is based on [Rasmussen, 2017], the introductory topology notes given to third semester students at Aalborg University.

2.1 Open and Closed Sets

We start out by defining a topology and will afterwards look at a few examples to help with understanding this abstract definition. We denote by $\mathcal{P}(X)$ the power set of a set X, i.e., $\mathcal{P}(X) = \{A \mid A \subset X\}$, the collection of all subsets of X. Note that $X, \emptyset \in \mathcal{P}(X)$.

Definition 2.1. Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$ be a collection of subsets of X satisfying the following.

(i) If I is any (not necessary countable) index set and $U_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

(ii) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

(iii) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

Then we call $(X, \mathcal{T})$ a *topological space* and $\mathcal{T}$ the *topology* of $(X, \mathcal{T})$. Additionally, we call elements of $\mathcal{T}$ *open sets* (in the topology $\mathcal{T}$).

Note that any sets X with at least two elements allow for multiple different topologies. We call the collection $\{\emptyset, X\}$ the trivial topology on X, and $\mathcal{P}(X)$ the discrete topology on X. Clearly, both are topologies (Why?).

Example 2.2. The following are examples of topological spaces. Take a moment for each of them to consider why.

(a) If $X = \{1, 2, 3, 4\}$, then the collection

$$\mathcal{T} = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$$

is a topology on X, and $(X, \mathcal{T})$ is a topological space.

(b) $(\mathbb{R}, \mathcal{T}_\mathbb{R})$ is a topological space, where

$$\mathcal{T}_\mathbb{R} = \{U \subset \mathbb{R} \mid U \text{ is open in the usual sense}\}.$$

This fact is a bit harder to prove, and is usually done after introducing the notion of a metric and a metric space, but we will not be needing metric spaces for our later work, so this will be omitted here.

(c) If X is any set (e.g. $X = \mathbb{R}$) then the *cofinite topology* on X is the collection $\mathcal{T}_{\text{co}} = \{U \subset X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Exercise. Find an example of a collection of open sets $U_i \in \mathcal{T}_\mathbb{R}$ for $i \in \mathbb{N}$ such that $\bigcap_{i=1}^{\infty} U_i \notin \mathcal{T}_\mathbb{R}$.

The above definition is abstract, and might be harder to grasp than alternative definitions. However, it is mathematically easier to manipulate, and thus often preferable. In general, we think of topologies as a way to understand "closeness" in a set. While the set X gives us the points of a space, the topology will allow us to determine how "close" two given points are. This is better understood after introducing the notion of neighbourhoods.

Definition 2.3. Let $(X, \mathcal{T})$ be a topological space and $x \in X$ a point. A *neighbourhood of x* is a subset $A \in \mathcal{P}(X)$ such that $x \in U \subset A$ for some $U \in \mathcal{T}$.

In this text we will denote by $\mathcal{N}(x)$ the collection of all neighbourhoods of $x \in X$. Note that an open neighbourhood of a point $x \in X$ would just be an open set containing x.

In terms of "closeness", we would say two points are "close" if they share "many" neighbourhoods.

Exercise. Show that if $U \in \mathcal{P}(X)$ is a neighbourhood of all $x \in U$, then U is open.

Proposition 2.4. *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The collection*

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$$

is called the trace topology on A, and forms a topology on A.

Proof. Exercise. □

The trace topology is also sometimes called the induced topology.

Definition 2.5. Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is called *closed* if $A^c = X \setminus A \in \mathcal{T}$.

Exercise. Show that

(a) $\emptyset$ and X are closed,

(b) any intersection of (not necessarily countably many) closed sets is a closed set, and

(c) any finite union of closed sets is a closed set.

The above result is an alternative definition of a topology via closed sets. It shows that we can describe topologies by defining either what the open sets are or what the closed sets are, for any given set X. The Zariski topology (the topology classically considered in algebraic geometry) is an important topology that is defined by its closed subsets.

Definition 2.6. Let X be a topological space and $A \subset X$.

(i) The *closure of* A is defined to be the intersection of all closed sets containing A. The closure is denoted $\overline{A}$.

(ii) The *interior of* A is defined to be the union of all open subsets contained in A. The interior is denoted A^o.

The definition is equivalent to $\overline{A}$ being the smallest closed subset containing A, and A^o being the biggest open subset contained in A. For any closed subset K and open subset U it holds that $\overline{K} = K$ and $U^o = U$ as expected. Note that if A is a neighbourhood of a point x, then $A^o \in \mathcal{N}(x)$.

We will in this text mostly work with topological spaces with a nice property called Hausdorff. This property insures that we can separate different points with neighbourhoods.

Definition 2.7. Let $(X, \mathcal{T})$ be a topological space. Two points $x, y \in X$ are said to be *separated by neighbourhoods* if there exists two neighbourhoods $N \in \mathcal{N}(x)$ and $M \in \mathcal{N}(y)$ such that $N \cap M = \emptyset$. If all pairs of distinct points in X are separated by neighbourhoods, then $(X, \mathcal{T})$ is called a Hausdorff space.

Exercise. Show that $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ is Hausdorff.

Exercise. Show that if X is infinite, then the space $(X, \mathcal{T}_{\text{co}})$, endowed with the cofinite topology, is not Hausdorff.

Hint: Show first that there exists no two non-empty disjoint open sets in $\mathcal{T}_{\text{co}}$.

2.2 Continuity

An important part of topology is the concept of continuity, and for topological spaces, the definition is simple.

Definition 2.8. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, and $f : X \to Y$ a function. f is said to be *continuous* (in $\mathcal{T}_X$ and T_Y) if $f^{-1}(V) \in \mathcal{T}_X$ for all $V \in \mathcal{T}_Y$.

Exercise. Show that $f : X \to Y$ is continuous if and only if $f^{-1}(A)$ is closed in X for all $A \subset Y$ closed in Y.

Exercise. Show that if $\mathcal{T}_Y = \{\emptyset, Y\}$ is the trivial topology, then all functions $f : X \to Y$ are continuous, for any space $(X, \mathcal{T}_X)$.

Exercise. Show that if $\mathcal{T}_X = \mathcal{P}(X)$ is the discrete topology, then all functions $f : X \to Y$ are continuous, for any space $(Y, \mathcal{T}_Y)$.

Exercise. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is continuous in the usual "ε-δ"-definition.

We usually distinguish between a function being continuous, and a function only being continuous at a point. We can make the same distinction for topological spaces.

Definition 2.9. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, $f : X \to Y$ a function, and $x \in X$ a point. f is said to be *continuous at the point* x if $f^{-1}(V) \in \mathcal{N}(x)$ for all $V \in \mathcal{N}(f(x))$, i.e., the pre-image of any neighbourhood of $f(x)$ is a neighbourhood of x.

Exercise. Show that $f : X \to Y$ is continuous if and only if f is continuous at all points of X.

A certain class of continuous functions allows us to define what it means for two topological spaces "to be the same". The "sameness" of topological spaces are often examplified by a coffee mug and a torus, since we intuitively understand that they, in some sense, are the same.

Definition 2.10. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, and $f : X \to Y$ a function. We call f a *homeomorphism* if it is continuous, bijective, and its inverse function $f^{-1} : Y \to X$ is continuous. If there exists a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, then the spaces are said to be *homeomorphic*.

If f is a homeomorphism whose codomain is the domain, f is called a self-homeomorphism.

Proposition 2.11. *Let $f : X \to Y$ be a homeomorphism. Show that*

(i) *$(X, \mathcal{T}_X)$ is Hausdorff if and only if $(Y, \mathcal{T}_Y)$ is Hausdorff,*

(ii) *f is an open mapping, i.e. $f(U) \in \mathcal{T}_Y$ if $U \in \mathcal{T}_X$, and*

(iii) *f is a closed mapping, i.e. $f(A)$ is closed if AX is closed.*

Proof. Exercise. □

Exercise. Show that if a function is continuous, bijective and an open mapping, then it is a homeomorphism.