Hodge Theory

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Abstract

Hodge Theory is an important part of Complex Algebraic Geometry. The goal of these seminars is to show the Hodge decomposition and the Lefschetz decomposition. Both of these decompositions represent special structures on the cohomology of Kähler manifolds.

1. Motivation

The first question that comes to mind is what is the Kähler manifold and why exactly this type of manifold we are interested in. Next theorem can give you more intuition about this (we will not go thought it during classes, but it is important to keep it in mind):

Theorem 1.1 (Kodaira)**.** A compact complex manifold admits a holomorphic embedding into a complex projective space if and only if it admits Kähler metric whose Kähler form is integral class.

So basically all smooth projective manifolds are special cases of Kähler ones. During seminars we will show precisely why a complex projective space is a Kähler manifold.

Hodge theory, named after William Vallance Douglas Hodge (1903–1975), puts the theory of partial differential equations to work to study the cohomology of smooth manifolds. The Hodge decomposition theorem, which lies at the heart of Hodge theory, uses these operators to decompose the space of differential k-forms into a direct sum of L_2-orthogonal subspaces.

2. Important

Please do not not be scared because of the topic it only seems to be complicated and heavy. It is not :)

It would be nice to know something extremely new and I'm totally open to all kinds of questions before or during classes. If it appears that the majority gets lost in the very beginning then we always can go slowly and precisely through everything that is unclear.

3. Preliminaries

Definition 3.1. A real (resp. complex) **topological vector bundle** of rank n over a topological space X is a topological space E and a projection map $\pi: E \to X$ such that it locally looks like $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$(resp. $U_i \times \mathbb{C}^n$). More rigorously would be to say that we have an open covering $\{U_i\}$ of X and homeomorphisms ϕ_i which are called "local trivializations"

$$\phi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n,$$

such that:

1. If $\mathrm{pr}: U_i \times \mathbb{R}^n \to U_i$ is just a projection then $\mathrm{pr} \circ \phi_i = \pi$
2. The transition functions $\phi_j \circ \phi_i^{-1}$ are $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) on each fibre $u \times \mathbb{R}^n$. So basically we can represent such transformation as $n \times n$ real matrix (resp. complex matrix).

A **section** of a vector bundle is a map $\sigma: X \to E$ such that $\pi \circ \sigma = id_X$. Almost always we will consider continuous or holomorphic sections.

A vector bundle said to be **trivial** if it admits a global trivializations $\phi: E \cong X \times \mathbb{R}^n$.

Remark 3.2. Notice that if we equip vector bundle in the defined above with a sheaf $\mathcal{F}$ such that $\mathcal{F}(U) = \{\text{continuous sections } \sigma: U \to E_{|U}\}$ then we will obtain exactly the sheaf of modules over the sheaf of real continuous functions (or holomorphic in colmex case). And this definition would be exactly the same as in scheme theory.

Also recall that we have $1:1$ correspondence between locally free sheaves of finite rank and vector bundles over some scheme X. I don't know why but for me it is more convenient to think about them in terms of locally free sheaves :)

Definition 3.3. Let X be a differential real manifold of dimension $2n$. Then we say that X is **equipped with a complex structure** if we can find an atlas (U_i, ϕ_i) such that U_i are diffeomorphic to open sets of $\mathbb{C}^n$ and transition maps $\phi_i \circ \phi_j^{-1}$ are holomorphic.

Remark 3.4. Note that when we are speaking about real manifolds we almost always interested in continuous or differential functions, but if we are speaking about complex ones then we often use holomorphic maps. As everyone knows holomorphic maps are extremely nice at least because they locally looks like analytical functions.

Definition 3.5. Let X be a complex manifold and (U_i, ϕ_i) its atlas. Then we can identify a tangent bundle $T_{U_i,\mathbb{R}}$ (standard real one) using differentials ϕ_i with $U_i \times \mathbb{C}^n$. So notice that we will have a natural isomorphism $T_{\mathbb{C}^n,x} \cong \mathbb{C}^n$ (in the same way as it was in real case).

The main point about tangent bundle is how it looks locally. Let we have a complex manifold X of dimension n. Suppose that we have a point x and real local coordinates around it are $x_1, \dots x_n, y_1, \dots y_n$. Then $T_{x,\mathbb{R}}$ is a $\mathbb{R}$-vector space with the basis given by $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$. Then we can define $z_i = x_i + iy_i$ and one can check that this is complex linear coordinates for X. Then we complexify it in a such a way that $T_{x,\mathbb{C}} = T_{x,X} \otimes \mathbb{C}$ and its basis looks like $\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z_k}}$.

Also since the representation described above we can decompose $T_{X,\mathbb{C}} = T^{1,0}_{X,\mathbb{C}} \oplus T^{0,1}_{X,\mathbb{C}}$ where $T^{1,0}_{X,\mathbb{C}}$ is a $\mathbb{C}$-vector space which is generated by basis $\frac{\partial}{\partial z_k}$ and $T^{0,1}_{X,\mathbb{C}}$ by $\frac{\partial}{\partial \overline{z_k}}$. Moreover we will have an isomorphism $T^{1,0}_{X,\mathbb{C}} \cong T_{X,\mathbb{R}}$ as $\mathbb{C}$-linear spaces. So in order to state it we need to endowed $T_{X,\mathbb{R}}$ with a complex structure. That can be done in a such a way that we put $i \cdot \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$.

Then we are go to the most important part a **vector bundle of differential forms** over a manifold X. It the most natural way it can be defines as a dual space for T_X and also it preserves decomposition $\Omega_{X,\mathbb{C}} = \Omega^{1,0}_{X,\mathbb{C}} \oplus \Omega^{0,1}_{X,\mathbb{C}}$ where $\Omega^{1,0}_{X,\mathbb{C}}$ is generated by dz_k and $\Omega^{0,1}_{X,\mathbb{C}}$ by $d\overline{z_k}$ locally. Notice that this is definition of 1-forms over a complex manifold X. In order to define k-form we just want to consider an exteriror algebra, i. e. $\Omega^k_{X,\mathbb{R}} = \bigwedge^k \Omega_{X,\mathbb{R}}$ in real case and

$$\Omega^k_{X,\mathbb{C}} = \bigoplus_{p+q=n} \Omega^{p,q}_{X,\mathbb{C}} = \bigoplus_{p+q=n} \bigwedge^p \Omega^{1,0}_{X,\mathbb{C}} \otimes \bigwedge^q \Omega^{0,1}_{X,\mathbb{C}}$$

So basically each k-differential is locally of the form:

$$\omega = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\ldots i_k} dz_{i_1} \wedge dz_2 \wedge \cdots \wedge d\overline{z_{i_k}}$$

Definition 3.6. The **exterior algebra** of a vector space V over a field is a space that admits an exterior product which is denoted by $\wedge: V \times V \to$ such that it satisfies the property $x \wedge x = 0$. From this follows, for example that we have that $x \wedge y = -y \wedge x$.

In order to define the **k-th exterior power of** V one can think about it as a vector space spanned by all expressions of the form $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ where e_i are basis vectors of a space V. With the same property described above.

Also note that we have well defined operator $d: \Omega^k_{X,\mathbb{R}} \to \Omega^{k+1}_{X,\mathbb{R}}$ in the following way. Assume that we have a k-form that locally is given by:

$$\omega = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\ldots i_k} dx_{i_1} \wedge dz_2 \wedge \cdots \wedge dx_{i_k}$$

Then we define $d\omega$ as a $k+1$-form that locally looks like:

$$d\omega = \sum_{m=1}^{n} \sum_{i_1<i_2<\cdots<i_k} \frac{\partial f_{i_1\ldots i_k}}{\partial x_m} dx_m \wedge dx_{i_1} \wedge dz_2 \wedge \cdots \wedge dx_{i_k}$$

Some basic definitions:

Definition 3.7. Let ω be a k-form, then we say that ω is a **closed** form if $d\omega = 0$. We say that form is **exact** if there exists a $k-1$-form α such that $\omega = d\alpha$.

Further one can find some problems to work out in order to understand better material provided above.

3.1. Show that one can provide a structure of complex manifold on $\mathbb{CP}^1$, i. e. write down precise charts especially near ∞.

3.2. Show using the maximum principle that a connected compact manifold possesses no holomorphic functions other then constant ones.

3.3. Show that $d^2 = 0$.

3.4. For a p-form α and a q-form β, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.

3.5 (Poincare lemma)**.** Show that any closed differential form on $\mathbb{R}^2$ is exact.