Constructing infinite sets

Benedek Nádor

Introduction

The method we are discussing, is rather practical so we introduce it through an example. Imagine we are asked to prove that there is a set S that has infinitely many elements, all integers greater than one. Furthermore, these integers should be pairwise coprime. Of course, we can solve it instantly by presenting a well-known set that satisfies the condition, namely the set of primes. But we don't need to specify explicitly the set; it's just enough to prove that there is one.

Construction Method

We start with the empty set T and insert the elements one by one. First, we make a list L of all integers greater than one. Then we choose any number a on this list (for the sake of simplicity the first one) and insert it into T. Then we erase all numbers from L that are not coprime to a. After this move, L contains infinitely many numbers with infinitely many different prime divisors, because in the previous steps, we only erased numbers divisible by a finite number of primes (we did this in each step and there were finitely many steps before the current one). So we can repeat this step infinitely times so S will contain infinitely many numbers that are pairwise coprime.

This solution is much more complicated than the first one. So why is it worth mentioning it? Often, a construction to a problem is not as straightforward as in the case of prime numbers. In those cases, we can use this method to prove that there is such a set and provide an algorithm to construct it.

Example

Is it possible to partition the set of integers into arithmetic progressions of 5 members such that the difference of each series is different?

Solution: Yes, it is possible. We show an algorithm that does the partitioning. First we push every integers into a list L. Let the set of the already-partitioned integers be S. In each step, we insert five numbers into S from L. In step i, we take the first number of L, l_i and the largest number of S, s_i. Let $d_i := s_i - l_i$. We insert $l_i, l_i + d_i, l_i + 2d_i, l_i + 3d_i, l_i + 4d_i$ into S and remove these elements from L. These numbers will form the ith quintuplet. They are different from the previous elements of S because l_i was not in S and the other elements are all larger than the greatest number in S. Throughout this algorithm, we assign each element of L into a quintuplet so we do the partitioning.

Problems

1. Is it possible to partition the set of integers into arithmetic progressions of 5 members such that the difference of each series is different and such that each difference $(1,2,3,4,...)$ occurs exactly once?

2. Is it true that coloring positive integers with 2 colors always results in an infinite arithmetic series of one color $(a, a+d, a+2d, ...)$?

3. A flea is jumping on the lattice points of the plane (point with integer coordinates). Can it move in a way that it visits each lattice point exactly once, the length of its jumps are positive integers, and each positive integer occurs exactly once among the lengths of the jumps?

4. We want to cover the plane (an infinite square grid in all directions) without gaps with all possible hole-free polyomino, without overlapping, using each polyomino exactly once. Is this possible?

5. (MBL 2022 qualification quiz) Is it possible to fill a $\mathbb{N} \times \mathbb{N}$ table (table with infinite length and height, but only in one direction) with all integers, without repetitions, such that each row is an increasing sequence and each column is a decreasing sequence?

6. For $H \subset \mathbb{Z}$ and $n \in \mathbb{Z}$ let h_n denote the number of finite subsets of H in which the sum of the numbers is n. Is there an $H \subset \mathbb{Z}$ for which $0 \notin H$ and for every $n \in \mathbb{Z}$ h_n is even (and finite)?

7. (Kürschák competition 2019) Is it true that if H and P are bounded subsets of $\mathbb{R}$, then there exists at most one set A such that $H = A+P = \{a+p : a \in A, p \in P\}$ and the sets $a+P$, $a \in A$ are disjoint?