# Problems with graphs and probabilities

### Problem 1

Let $G$ be a graph of $n$ vertices. Prove that the number of vertices with odd degree is even.

### Problem 2

In the following, we consider spanning subgraphs of graphs. That is subgraphs where we keep all vertices and some of the edges. Prove that the space of spanning subgraphs of the graph $G$ is a group with symmetric difference $\triangle$ as the group operation. We denote this group by $\mathcal{E}(G)$.

### Problem 3

Let $\Omega_\emptyset(G)$ be the set of even spanning subgraphs of a finite or (locally finite) infinite graph $G$. Prove that $\Omega_\emptyset(G)$ is a subgroup of $\mathcal{E}(G)$.

### Problem 4

Bernoulli percolation $\mathbb{P}_p$ with parameter $p \in [0,1]$ is a probability measure on $\mathcal{E}(G)$. Consider the example of the graph $\mathbb{Z}$ with vertices $v \in \mathbb{Z}$ and edges $(v, v+1)$ for each $v \in \mathbb{Z}$. Compute $\mathbb{P}_p[0 \leftrightarrow n]$. We let $0 \leftrightarrow \infty$ be the event that 0 is part of an infinite path. Compute $\mathbb{P}_p[0 \leftrightarrow \infty]$.

### Problem 5

What is the Haar measure on $\mathcal{E}(G)$? Let $\mu$ be the Haar measure of the group $\Omega_\emptyset(\mathbb{Z})$. Compute $\mu[0 \leftrightarrow \infty]$.

### Problem 6

We consider the periodic Manhattan lattice with alternating orientations going north-south and east-west. Place obstructions on vertices independently with probability $0 < p < 1$. A particle is moving on the edges with unit speed following the orientation of the lattice and it will turn only when encountering an obstruction. The problem is that for which value of p is the trajectory of the particle closed almost surely.

### Problem 7

Consider the hypercubic lattice $\mathbb{Z}^d$. Label every edge with the outcome of uniform independent random variables $U_e \sim \text{Unif}[0,1]$. We start a process at 0 where we iteratively add the edge with the lowest number to the connected component. Thereby we construct a random infinite graph $\mathbb{G}$. If $B_R$ is the set of edges within graph distance $R$ of 0. Prove that $\frac{|B_R \cap \mathbb{G}|}{R^d} \to 0$ almost surely.