Holistic introduction to $D^b(X)$

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Abstract

For a projective variety X, we use $D^b(X)$ to denote the derived category of coherent sheaves on X. During this talk, I will give a holistic introduction and try to convey the importance of this category.

Imagine that you are working on a geometry problem in the plane (so over $\mathbb{R}$). You are looking at two circles which do not intersect. Using your knowledge of complex numbers, you can draw the line through the two complex intersection points between the circles (this is the radical axis) although you cannot see the two intersection points. You can even invert in one of the complex intersection points of the two circles. In other words, you can take the geometry problem over $\mathbb{R}$ and go up to the world of geometry over $\mathbb{C}$ where you can see so much more, find a solution, and then go down with your solution back to the world of geometry over $\mathbb{R}$ again.

Looking at derived categories, is like looking at a whole new class of geometry which is much greater than the geometry over $\mathbb{C}$. An exceptional object in $D^b(X)$ is just like a point on X which you can't see when you just work over $\mathbb{C}$. Just the same way that you can't see the tangent points between two concentric circles if you work over $\mathbb{R}$. Using explicit examples I will try to give you some holistic understanding of the importance of derived categories in algebraic geometry.