Complex bash

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1 Complex numbers (warm-up)

$$z = a + ib = re^{i\theta}$$

with

$$i^2 = -1$$

where (conversion by *Euler identity*)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Sidequest: derive conversion between $(a, b) \rightsquigarrow (r, \theta)$
Each complex number has its complex conjugate (will be very useful, as then we can speak in only complex numbers)

$$\bar{z} \equiv z^* = a - ib = re^{-i\theta}$$

For the sake of communication, the following notation is used:

$$\Re(z) \coloneqq a; \quad \Im(z) \coloneqq b; \quad |z| \coloneqq r; \quad \arg z \coloneqq \theta$$

facts to prove:

- $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$
- $(z_1 z_2)^* = z_1^* z_2^*$
- $(1/z)^* = 1/z^*$
- $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

2 Geometry (pro-tip: only lazy solves)

We mark points of the place with their corresponding complex number (a being the x coordinate, b being the y one) and name them by the corresponding non-Capital letter (number o for point O, etc.)

- Take a point z on the unit circle ($|z| = 1$). Then $z^* = \frac{1}{z}$.
- $zz^* = |z|^2$ (way to measure (equality of) lengths)

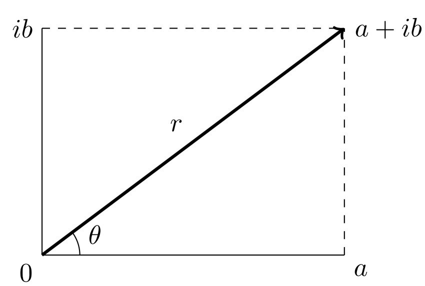

Figure 1: Different complex number interpretations

- $\frac{z}{z^*} = e^{2i\arg z}$(way to measure (equality of) angles)
- Particular cases for angles: $e^{2i\pi} = 1, e^{i\pi} = -1$
- A segment AB is very well represented by $b - a$.
- Length of a segment AB:

$$|AB|^2 = (a - b)(a^* - b^*)$$

- Parallel lines AB and CD by equating their directions:

$$\frac{a - b}{a^* - b^*} = \frac{c - d}{c^* - d^*}$$

- Claim that $AB \parallel AC$ means A, B, C being colinear.
- Perpendicular lines AB and CD by noting their directions opposite by sign:

$$\frac{a - b}{a^* - b^*} = -\frac{c - d}{c^* - d^*}$$

- Useful identity, when you have B, C on the unit circle:

$$\frac{b - c}{b^* - c^*} = -bc$$

- Homothety (ratios) work nicely: midpoint of AB is
- Mass centre of a triangle ABC
- Orthocentre H of ABC:

3 Problems (feel free to later try synthetically as well)

1. Derive an expression for the intersection of two chords AB and CD of the unit circle.
2. Let H be the orthocentre of triangle ABC. Show that its reflections over BC (called R_1) and over the midpoint of BC (called R_2) both lie on the circumcircle
3. Prove Euler's line (O, H, G are colinear)
4. P and Q are on the unit circle. Derive the expression for the intersection of the tangents through P and Q (also called the polar of PQ).
5. Let AC be the diameter of a circle, containing points A, B, C, D. Let X the intersection of AB and CD, Y be the intersection of tangents through B and D. Prove that $|AC| = |XY|$.
6. Let D, E be feet of altitudes from A, B of triangle ABC, H its orthocentre. DE intersects AB at K. Prove that the median $CM \perp HK$.
7. Derive the expression for the feet of angle bisectors D, E, F of triangle ABC on the unit circle; the expression for the incentre. (think of smarter ways to do that)
8. [Napoleonian triangles] Let ABC be a triangle. On its three sides we erect equilateral triangles ABF, BCD, CAE. Prove that the centres of these triangles are the vertices of an equilateral triangle.
9. [IMO 2009] Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.
10. [IMO 2012] Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.