

Functional equations

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1. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + y) = f(x) + y$$

for all $x, y \in \mathbb{R}$

2. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x - y) = f(x) - 2xy + f(y)$$

for all $x, y \in \mathbb{R}$

3. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{Q}$

4. (SMT-finals 2018). Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) + 2f(\sqrt[3]{1 - x^3}) = x^3$$

for all $x \in \mathbb{R}$

5. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x)f(y) = f(xy) + x + y$$

for all $x, y \in \mathbb{R}$

6. Find all polynomials $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$xp(x - 1) = (x - 2)p(x)$$

for all $x \in \mathbb{R}$

7. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + y) = f(x) + 2xy + f(y)$$

for all $x, y \in \mathbb{R}$

(Is it possible to solve without continuity? I've got no idea)

8. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that

$$f(1) = 2 \text{ and } f(xy) = f(x)f(y) - f(x + y) + 1$$

for all $x, y \in \mathbb{Q}$

9. (USAMO 2002). Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all $x, y \in \mathbb{R}$

10. (IMO, 1968). Let $f(x)$ be a real-valued function defined for all real numbers x , such that for some positive constant a the equation

$$f(x + a) = \frac{1}{2} + \sqrt{f(x) - f^2(x)} \text{ holds for all } x \in \mathbb{R}.$$

Prove that $f(x)$ is periodic, and, for $a = 1$, give an example of such a non-constant function $f(x)$.