Some properties of Triangulated Categories
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1 Defining a Triangulated Category

I will not define additive categories and some other useful definitions, but if you have any questions about additive categories or categories in general, feel free to ask. (That also holds for triangulated categories).

First, some motivation for triangulated categories. The main problem with any motivation is that the fourth axiom especially is quite contrived, which makes showing that an example is indeed an example of a triangulated category quite contrived. However, some of the examples are quite interesting, the derived category being one of them. Some other examples are vector spaces over a field, the homotopy category and stable module category. Another reason it is interesting is that most categories that are abelian are not triangulated, as being both is extremely restrictive. Hence we get useful information on nonabelian categories, and we get a notion that is similar to exactness in distinguished triangles. However, this class will still mostly be based on using definitions to solve problems, and not based on some intuitive sense of what the objects are.

Definition 1.1. (1) An *additive functor* Σ is a functor between additive categories $\mathcal{A}, \mathcal{B}$ where the induced map between the abelian groups $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(\Sigma X, \Sigma Y)$ is a group homomorphism for all pairs of objects $X \in \mathcal{A}, Y \in \mathcal{A}$.

(2) If the additive functor is an automorphism we call it a *shift functor*.

Remark 1.2. (1) In particular, we have that any shift functor Σ has an inverse functor Σ^{-1}.

(2) Some authors prefer defining the shift functor as an auto-equivalence instead. This is somewhat weaker, but still yields the same results.

Some examples of shift functors are the identity functor or the functor moving a complex 1 place to the left. (A complex is a chain of objects and morphisms such that the composition of two morphisms is zero).

We will now define triangles and morphisms between them.

Definition 1.3. (1) A *triangle* is a diagram in an additive category on the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

where the functor Σ is a shift functor.

(2) A *morphism of triangles* between two triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

is a collection of morphisms $f : X \to X', g : Y \to Y', h : Z \to Z'$ such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

Remark 1.4. One may compose morphisms of triangles by composing the morphisms elementwise.

Since we have a morphism of triangles from a triangle to itself consisting of the identity morphisms, one may therefore define isomorphisms of triangles.

Definition 1.5. We say that a morphism between triangles is an *isomorphism between triangles* if there exists an inverse morphism of triangles. That is, there exists a morphism of triangles such that their composition both ways is the morphism of triangles consisting of the identity morphisms. Note that this is equivalent to the morphisms f, g, h being isomorphisms.

We will now look at the definition of a triangulated category, where triangles are essential. The essence of a triangulated category is a set of distinguished triangles, which behave similarly to exact sequences.

Definition 1.6. A *triangulated category* is a additive category $\mathcal{A}$ with a shift functor Σ and a collection of triangles called *distinguished triangles* that satisfy the following axioms.

TR1 a) Any triangle that is isomorphic to a distinguished triangle is also a distinguished triangle.

b) $\forall X \in \mathcal{O}\mathfrak{b}(\mathcal{C})$, the following triangle is distinguished

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \Sigma X$$

c) $\forall X, Y \in \mathcal{O}\mathfrak{b}(\mathcal{C})$, and $f \in \mathrm{Hom}(X, Y)$, we have a distinguished triangle on the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

TR2 The two following triangles below are distinguished if and only if the other triangle is distinguished

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

TR3 Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$$

and morphisms $f : X \to X', g : Y \to Y'$, such that $gu = u'f'$, then there exists a morphism $h : Z \to Z'$, such that the diagram below is a morphism of triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

TR4 Given distinguished triangles

$$X \xrightarrow{u_1} Y \xrightarrow{v_1} Z' \xrightarrow{w_1} \Sigma X$$

$$Y \xrightarrow{u_2} Z \xrightarrow{v_2} X' \xrightarrow{w_2} \Sigma Y$$

$$X \xrightarrow{u_2 u_1} Z \xrightarrow{v_3} Y' \xrightarrow{w_3} \Sigma X$$

there exists a distinguished triangle

$$Z' \xrightarrow{u_4} Y' \xrightarrow{v_4} X' \xrightarrow{w_4} \Sigma Z'$$

making the following diagram commute:

$$\begin{array}{ccccccc}
X & \xrightarrow{u_1} & Y & \xrightarrow{v_1} & Z' & \xrightarrow{w_1} & \Sigma X \\
\downarrow 1_X & & \downarrow u_2 & & \downarrow u_4 & & \downarrow 1_{\Sigma X} \\
X & \xrightarrow{u_2 u_1} & Z & \xrightarrow{v_3} & Y' & \xrightarrow{w_3} & \Sigma X \\
\downarrow u_1 & & \downarrow 1_Z & & \downarrow v_4 & & \downarrow \Sigma u_1 \\
Y & \xrightarrow{u_2} & Z & \xrightarrow{v_2} & X' & \xrightarrow{w_2} & \Sigma Y \\
\downarrow v_1 & & \downarrow v_3 & & \downarrow 1_{X'} & & \downarrow \Sigma v_1 \\
Z' & \xrightarrow{u_4} & Y' & \xrightarrow{v_4} & X' & \xrightarrow{w_4} & \Sigma Z'
\end{array}$$

Remark 1.7. (1) Axiom **TR3** is unnecessary, as it follows from the others. It is however included in most definitions of a triangulated category. It is also quite annoying to prove that it follows from the others, and it is very useful.

(2) One can replace **TR3** and **TR4** with Neeman's mapping cone axiom. The axiom states that if the triangles below are distinguished

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$$

and there exist morphisms $f : X \to X', g : Y \to Y'$, such that $gu = u'f$, then there exists a morphism $h : Z \to Z'$, such that the diagram below is a morphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow f & & \downarrow g & & h\downarrow & & \downarrow \Sigma f \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}$$

and that the cone of the morphism of triangles is distinguished:

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

(3) Axioms **T2** and **T4** together also imply that we have distinguished triangles on the form

$$\Sigma^{-1} Z \longrightarrow X \xrightarrow{f} Y \longrightarrow Z$$

$$\Sigma^{-1} Y \longrightarrow \Sigma^{-1} Z \longrightarrow X \xrightarrow{f} Y$$

given a morphism $f : X \to Y$.

In the examples mentioned earlier, we have that following sets of distinguished triangles:

In the case of vector spaces over a field the distinguished triangles are exact sequences that repeat consisting of three objects

In the case of the homotopy category and the derived category they consist of triangles isomorphic to the tirangles where the third object is the cone of the first morphism.

2 Problems

Now with most of the definitions already introduced, we will look at some problems.

Problem 2.1. Let the following triangle be distinguished:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

Show that $v \circ u = 0$, $w \circ v = 0$ and $\Sigma u \circ w = 0$.

Problem 2.2. Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

and the two morphisms $g : Y \to Y', h : Z \to Z'$ such that the diagram below commutes. Show that there exists a morphism $f : X \to X'$ making the diagram a morphism of triangles.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ & & \downarrow{g} & & \downarrow{h} & & \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

Remark 2.3. This obviously then also holds when we have f, h and need g.

Definition. We say that a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is *exact* if for all $T \in \mathcal{T}$, the images of the functors $\mathrm{Hom}(T, -)$, $\mathrm{Hom}(-, T)$ applied on the triangle are long exact. That is, the sequences

$$\cdots \xrightarrow{(\Sigma^{-1}w)_*} \mathrm{Hom}(T, X) \xrightarrow{u_*} \mathrm{Hom}(T, X) \xrightarrow{v_*} \mathrm{Hom}(T, X) \xrightarrow{w_*} \mathrm{Hom}(T, \Sigma X) \xrightarrow{(\Sigma u)_*} \cdots$$

$$\cdots \xrightarrow{(\Sigma u)^*} \mathrm{Hom}(\Sigma X, T) \xrightarrow{w^*} \mathrm{Hom}(Z, T) \xrightarrow{v^*} \mathrm{Hom}(Y, T) \xrightarrow{u^*} \mathrm{Hom}(X, T) \xrightarrow{(\Sigma^{-1}w)^*} \cdots$$

are long exact. Here the notation u_* means $\mathrm{Hom}(T, u)$ and $u^* = \mathrm{Hom}(u, T)$.

Problem 2.4. Let $\mathcal{T}$ be a triangulated category, and the following be a distinguished triangle:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

Furthermore let $T \in \mathcal{T}$. Show that the following induced sequences of abelian groups are exact:

$$\cdots \xrightarrow{(\Sigma^{-1}w)_*} \mathrm{Hom}(T,X) \xrightarrow{u_*} \mathrm{Hom}(T,Y) \xrightarrow{v_*} \mathrm{Hom}(T,Z) \xrightarrow{w_*} \mathrm{Hom}(T,\Sigma X) \xrightarrow{(\Sigma u)_*} \cdots$$

$$\cdots \xrightarrow{(\Sigma u)^*} \mathrm{Hom}(\Sigma X,T) \xrightarrow{w^*} \mathrm{Hom}(Z,T) \xrightarrow{v^*} \mathrm{Hom}(Y,T) \xrightarrow{u^*} \mathrm{Hom}(X,T) \xrightarrow{(\Sigma^{-1}w)^*} \cdots$$

That is, all distinguished triangles are exact triangles.

Problem 2.5. Let the following triangles be exact triangles:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

Show that if f, g are isomorphisms in the commutative diagram below, then h is also an isomorphism.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

Remark 2.6. By **T3** we may also conclude that f or g are isomorphisms given that g, h or f, h are isomorphisms respectively.

Problem 2.7. Let the following two triangles be distinguised:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$$

Show that they are isomorphic as triangles.

Lemma 2.8. *Let $\mathcal{C}_1, \mathcal{C}_2$ be additive categories, and $F : \mathcal{C}_1 \to \mathcal{C}_2$ an additive functor. Then we have that $F(A \oplus B) = F(A) \oplus F(B)$ for all $A, B \in \mathcal{C}_1$, where $i_{F(A)} = F(i_A), \pi_{F(A)} = F(\pi_A), i_{F(B)} = F(i_B), \pi_{F(B)} = F(\pi_B)$. Furthermore we have that $F(0) = 0$.*

Remark. In particular, this holds when $\mathcal{C}_1$ is a triangulated category, and the functor F is the functor $\mathrm{Hom}(T, -)$ for some $T \in \mathcal{C}$.

Problem 2.9. Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

be distinguished triangles. Show that their direct sum is distinguished.

Problem 2.10. Let

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

be a distinguished triangle, and let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle such that the following diagram commutes

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_Z} & & \downarrow{\scriptstyle \Sigma i_X} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \\
\downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \Sigma \pi_X} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X
\end{array}$$

and $\pi_X \circ i_X = 1_X, \pi_Y \circ i_Y = 1_Y, \pi_Z \circ i_Z = 1_Z$. Show that the second triangle is also distinguished.

Problem 2.11. Let the following triangle be distinguished:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

Show that the following four statements are equivalent:

1) w=0.

2) There exists a morphism $u' : Y \to X$ such that $u' \circ u = 1_X$.

3) There exists a morphism $v' : Z \to Y$ such that $v \circ v' = 1_Z$.

4) There exist morphisms $u' : Y \to X, v' : Z \to Y$ such that $u \circ u' + v' \circ v = 1_Y$, $u' \circ u = 1_X, v \circ v' = 1_Z$. That is $Y \simeq X \oplus Z$, with u, u', v, v' as projection and inclusion maps.

Problem 2.12. Show that the existence of morphisms u', v' such that $u \circ u' + v' \circ v = 1_Y$ is not equivalent to the statements in the theorem.

Problem 2.13. a) Show that $u : X \to Y$ being a monomorphism is equivalent with it being a split monomorphism.

b) Show that this does not hold in a general category.

Problem 2.14. Show that if $\mathcal{A}$ is an abelian category and

$$0 \xrightarrow{0} X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{0} 0$$

is a short exact sequence, then

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is distinguished in the derived category for some morphism w. Here X, Y, Z denote the complexes in the derived category with the objects X, Y, Z in the zeroth position respectively.