

The Anchor Point Lemma

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1 Theory

The Anchor Point Lemma is a series of configurations surrounding circles passing through D with well-defined intersections with sides of a triangle.

1. **(Anchor Point Lemma)** Given a triangle $\triangle ABC$, let D be an arbitrary point on BC , then let DE and DF be parallel to AC and AB respectively. Let (AEF) intersect (ABC) at G , let GD intersect (ABC) at X . Prove, that if M is the midpoint of AC , then A, X and M are colinear.
2. **(Generalized Anchor Point Lemma)** Let D be an arbitrary point on BC in $\triangle ABC$. Let there be two fixed directions l_1 and l_2 . Let E and F be the intersection of two lines through D parallel to l_1 and l_2 with AB and AC . Let G be the intersection of (AFE) and (ABC) . Prove that DG passes through a constant point X on (ABC) .

Also prove that the circles (AFE) are all coaxial and the second constant point is the intersection of the circle with the line isogonal to AX in $\angle CAB$ with (AFE) .

Remark 1.1 The circle (AGD) passes through a fixed point K on BC such that AK is isogonal to AX . If $P = CE \cap BF$ then the line DP passes through a fixed point.

3. **(Advanced Anchor Point Lemma)** Let P be an arbitrary fixed point and X an arbitrary fixed point on (ABC) . Let D be an arbitrary point on BC . Let PD intersect AC at E . Let XD intersect (ABC) at G . Let (AEG) intersect AB a second time at F . Prove that the line DF passes through a constant point Q as D moves on BC , and that (AEF) passes through a fixed point W .

Remark 1.2 Similarly to the previous lemma if $P = CE \cap BF$ then the line DP passes through a fixed point.

Note: More details about this method can be found at <https://mrmineev.com/articles/anchor-point/main.html>

2 More Advanced Theory

Let $\text{Anchor}_P^{\triangle ABC}(E)$ be the Anchor Point conjugate of point P where E is the parameter. Notice,

1. $\text{Anchor}_E^{\triangle ABC}(AC)$ is a line which passes through C and the intersection of AE with the line through B parallel to AC .
2. $\text{Anchor}_E^{\triangle ABC}(AB)$ is a line which passes through B and the intersection of AE with the line through C parallel to AB .

3. $\text{Anchor}_{(ABC)}^{\triangle ABC}(P)$ is a line parallel to AC which passes through the intersection of the line through P parallel to AB with BC .
4. Let \mathcal{L} be an arbitrary line in the plane then $\text{Anchor}_E^{\triangle ABC}(\mathcal{L})$ is a line which passes through $\mathcal{L} \cap BC$.

In addition to these properties the cross-ratio is preserved.

3 Problems

1. **(The Shooting Lemma)** Consider the chord BC in the circle Ω . Let the circle ω touch BC at a point D and the circle Ω at a point E . Prove that the line DE passes through M , the middle of the larger arc \widehat{BC} .
2. **(Sharky-Devil Config)** Let (I) be the incircle of $\triangle ABC$, let D, E, F be the tangency points of (I) with BC, AB and AC , respectively. Let G be the second intersection of (AFE) with (ABC) . Let S be the midpoint of the arc BC . Prove that S, D and G are colinear.
3. **(USA TST 2008 P7)** Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC . Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC , the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.
4. **(USA TST 2012 P1)** In acute triangle ABC , $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC . Points D and E lie on sides AB and AC , respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC , the circumcircle of triangle ADE passes through a fixed point other than A .
5. **(ELMO 2013 Shortlist G3)** Given $\triangle ABC$, a point D lies on line BC . The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A , and that this point lies on the median from A to BC .
6. **(APMO 2012 P4)** Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC , by M the midpoint of BC , and by H the orthocenter of ABC . Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH , and F be the point of intersection (other than E) of the line ED and the circle Γ . Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.
7. **(based on Moscow Mathematical Olympiad 2015)** Let $\triangle ABC$ be an isosceles triangle. Let P and Q be points on sides AB and AC , respectively, such that $CP = AQ$. Let X be a point on BC such that $PX = PC$ and T be the second intersection of (APQ) with (ABC) . Prove that $\angle ATX = 90^\circ$.
8. **(IMO Shortlist 2016 G2)** Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of \overline{BC} . The points D, E, F are selected on sides $\overline{BC}, \overline{CA}, \overline{AB}$ such that $ID \perp BC, IE \perp AI$, and $IF \perp AI$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A . Prove that lines XD and AM meet on Γ .
9. **(BMO Shortlist 2017 G5)** Let ABC be an acute angled triangle with orthocenter H . centroid G and circumcircle ω . Let D and M respectively be the intersection of lines AH and AG with side BC . Rays MH and DG intersect ω again at P and Q respectively. Prove that PD and QM intersect on ω .

10. (APMO 2022 P2) Let ABC be a right triangle with $\angle B = 90^\circ$. Point D lies on the line CB such that B is between D and C . Let E be the midpoint of AD and let F be the second intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.

Let X be a point on the arc AB of Γ not containing C , such that $\angle AXH = \angle AFH$. Let K be the circumcenter of triangle XIA . Prove that the lines AO and KI meet on Γ .

11. (IMO Shortlist 2005 G5) Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC , and let M be the midpoint of the side BC . Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.
12. (by @SBYT) Given a triangle $\triangle ABC$ with an incenter I , let D be a point on segment BC . Let the perpendicular line from D to CI meet BI at E and the perpendicular line from D to BI meet CI at F . Let the perpendicular line from D to BC meet EF at G . Prove that the circles (ABC) and the circle centered at G with radius GD are tangent.

4 Some of my Own

- (AoPS) Let D be an arbitrary point on the side BC of triangle $\triangle ABC$. Let E and F be the intersections of the lines through D , parallel to AC and AB , with AB and AC , respectively. Let G be the intersection point of the circumcircle of triangle AFE with the circumcircle of triangle ABC . Let M be the midpoint of BC , and let X be the second intersection point of line AM with the circumcircle of triangle ABC . Prove that lines EF, AD , and the tangent to the circumcircle of triangle XMG at point M are concurrent.
- (AoPS) Let D be an arbitrary point on the side BC in a triangle $\triangle ABC$. Let E and F be the intersection of the lines parallel to AC and AB through D with AB and AC . Let G be the intersection of (AFE) with (ABC) . Let M be the midpoint of BC and X the intersection of AM with (ABC) . Let H be the intersection of (XMG) with BC . Prove that EF is parallel to AH .
- (AoPS) Let D be an arbitrary point on the side BC in a triangle $\triangle ABC$. Let E and F be the intersection of the lines parallel to AC and AB through D with AB and AC . Let G be the intersection of (AFE) with (ABC) . Let M be the midpoint of BC and X the intersection of AM with (ABC) . Let J be the intersection of (XFG) with AC . Prove that XB, AD and JM are concurrent at P .

5 A bit more theory...

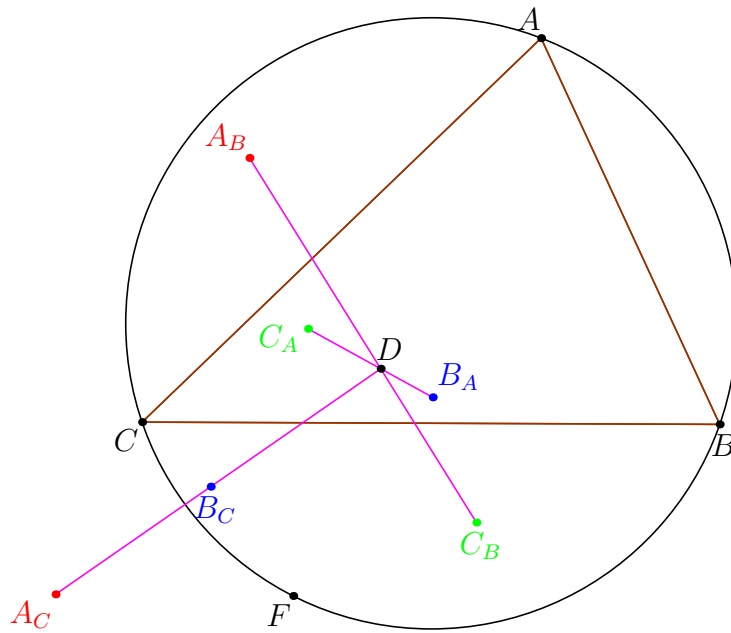
The Anchor Point Conjugate Theorem can be reformulated as,

Theorem 5.1 Fix a point $F \in (ABC)$ and an arbitrary point D in the plane. Let ω be a circle passing through A and D . Let X and Y be the second intersection points of ω with AC and AB , respectively. Let Q be the second intersection of ω with the circumcircle (ABC) , and define

$$P = FQ \cap BC.$$

Then, as ω varies, the lines PX and PY each pass through a fixed point.

Let these two points be called the *Evil Anchor Point* conjugates of D in $\triangle ABC$ with respect to F , which will be denoted as $\text{Evil}_{F,W}^{\triangle ABC}(D)$ where W is either B or C depending on whether it is the point which lies on PX or PY .



1. **(Evil Characterization)** Let (ABD) meet lines AC and AF at X and Y , respectively, then $\text{Evil}_{F,B}^{\triangle ABC}(D) = BX \cap DY$.
2. **(AoPS)** Prove that,

$$\begin{cases} D \in \text{Evil}_{F,B}^{\triangle ABC}(D) \cap \text{Evil}_{F,B}^{\triangle CBA}(D) \\ D \in \text{Evil}_{F,A}^{\triangle BAC}(D) \cap \text{Evil}_{F,A}^{\triangle CAB}(D) \\ D \in \text{Evil}_{F,C}^{\triangle ABC}(D) \cap \text{Evil}_{F,C}^{\triangle BAC}(D) \end{cases}$$

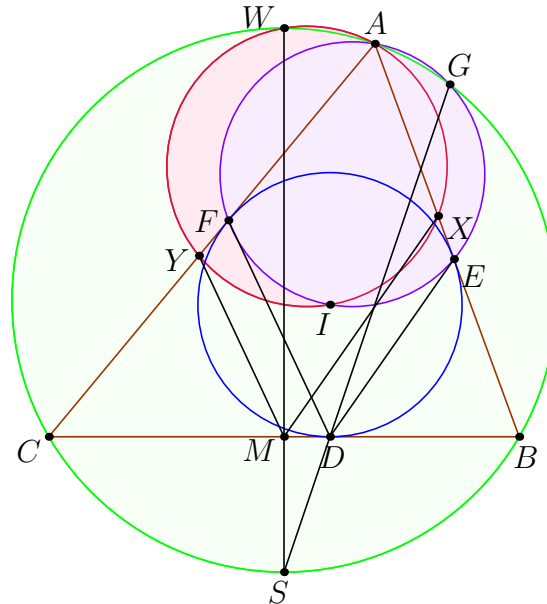
6 Extra (Unsolved)

1. **(USEMO 2025 P2)** Let ABC be a fixed triangle with circumcircle ω . Consider P a variable point inside ABC . Ray BP meets side AC at Y while ray CP meets side AB at X . Let Q be the second intersection of ω and the circumcircle of triangle AXY . Let K be the second intersection of ray AP and ω .
 Prove that as P varies, the circumcircles of triangle QPK all have a common radical center.

7 Proposed Solutions to Some Problems

Problem 7.1 (Sharky-Devil Configuration)

Let (I) be the incircle of $\triangle ABC$, let D, E, F be the tangency points of (I) with BC, AB and AC , respectively. Let G be the second intersection of (AFE) with (ABC) . Let S be the midpoint of the arc BC . Prove that S, D and G are colinear.



Proof. Let us introduce M the midpoint of BC and W the midpoint of the larger arc BC . Then, let X and Y be the intersections of the lines through M parallel to DE and DF with AB and AC , respectively. By the Generalized Anchor Point Lemma all that is left to prove is that $AWXY$ is cyclic.

Notice, since,

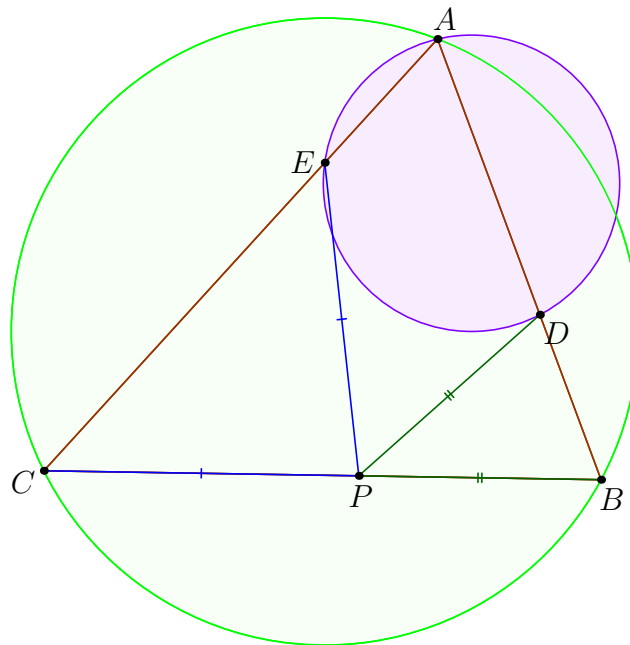
$$\angle WXY = \angle WAC$$

$$\angle XYW = \angle XAW$$

which implies that $\angle WAC = 180 - \angle BAW$, however this is only true for W being the midpoint of the larger arc BC . Thus $WAXY$ is cyclic which proves one of the properties of the Sharky-Devil point. (Amusingly I lies on this circle as well due to $I \in (AFE)$ and I lying on the angle bisector of $\angle CAB$). ■

Problem 7.2 (USA TST 2012 P1)

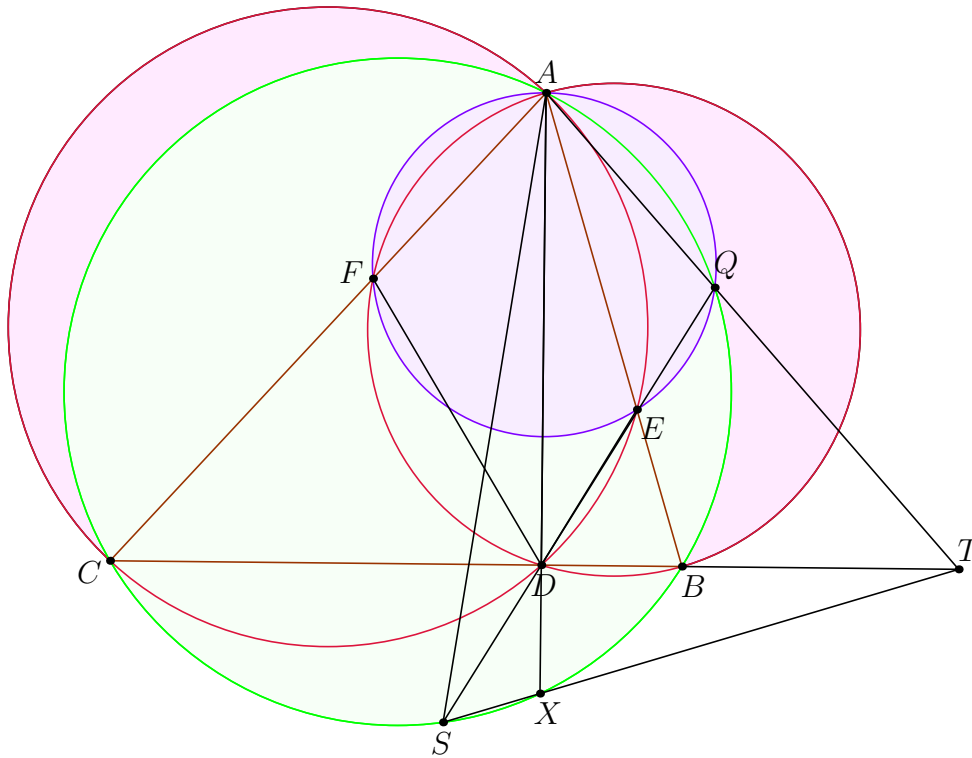
In acute triangle ABC , $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC . Points D and E lie on sides AB and AC , respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC , the circumcircle of triangle ADE passes through a fixed point other than A .



Proof. Since as we move P the lines EP and PD are parallel to two fixed directions, thus by the Generalized Anchor Point Lemma it must be that (AED) passes through a fixed point. ■

Problem 7.3 (ELMO 2013 Shortlist G3)

In $\triangle ABC$, a point D lies on line BC . The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A , and that this point lies on the median from A to BC .



Proof. Notice, FD and DE point in constant directions, since $\angle CDF = \angle A = \angle EDB$. Thus, by the Generalized Anchor Point Lemma all we need to do is show for one position of D that (AFE) passes through some fixed point on the median. Let us fix D to be the foot of the altitude from A to BC . Let S be the intersection of the symmedian from A with (ABC) , then, (it is well known, however the proof is outlined below)

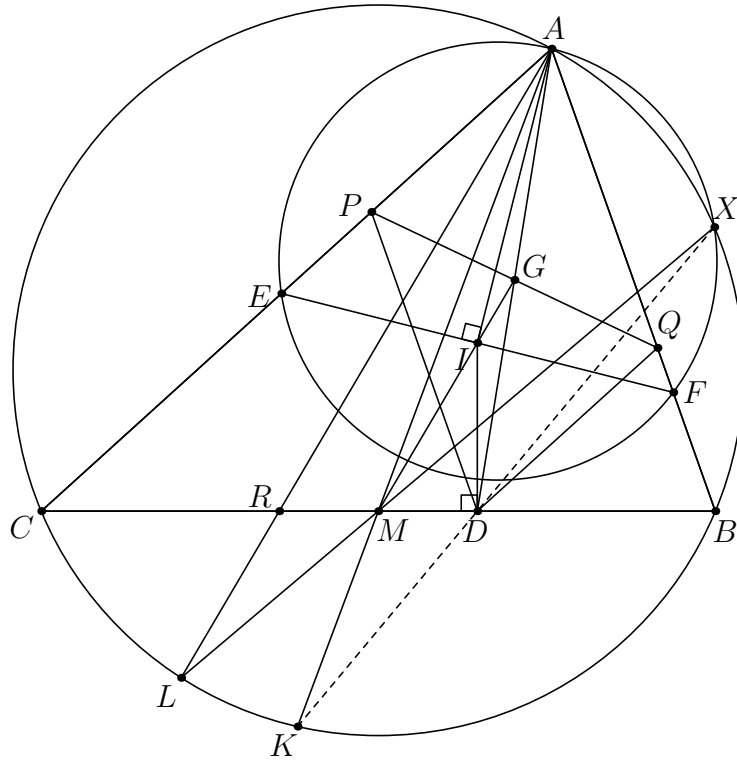
Lemma 7.1 S, D, Q are colinear.

indeed, since $ABSC$ is harmonic, by projecting from T it must be that $QBCX$ is harmonic, consequently projecting from D we obtain that Q goes to a point W on (ABC) such that $ABCW$ is harmonic, thus $W = S$, thus Q, D, S are colinear. \square

Now, by the Generalized Anchor Point Lemma since S, D, Q are colinear, it must be that (AEF) passes through a fixed point lying on the isogonal line to AS in $\angle CAB$ which is the median. \blacksquare

Problem 7.4 (IMO Shortlist 2016 G2)

Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of \overline{BC} . The points D, E, F are selected on sides $\overline{BC}, \overline{CA}, \overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A . Prove that lines XD and AM meet on Γ .



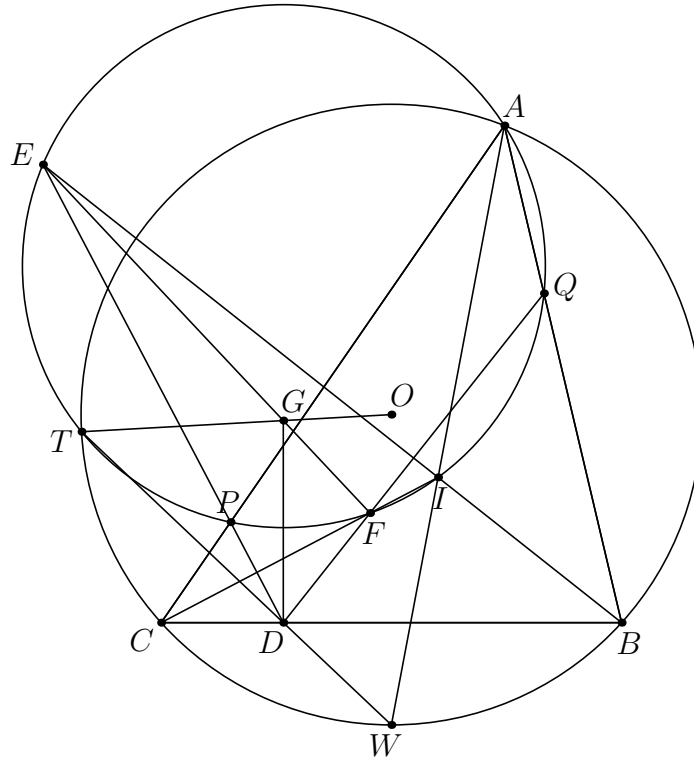
Proof. Let $K = AM \cap (ABC)$. Let P and Q be the intersections of parallel lines through D with respect to AB and AC with AC and AB , respectively.

Let $L = MX \cap (ABC)$, $R = AL \cap BC$ and $G = PQ \cap AD$. From the Anchor Point lemma we know that $GM \parallel AL$, however it is a well known lemma that M, I and G are collinear. Thus we see that $MI \parallel AR$ and the reflection of D over I lies on AR , consequently it must be that M is the midpoint of RD .

Concluding by Butterfly theorem we see that K, D and X are collinear. ■

Problem 7.5 (SBYT)

Given a triangle $\triangle ABC$ with an incenter I , let D be a point on segment BC . Let the perpendicular line from D to CI meet BI at E and the perpendicular line from D to BI meet CI at F . Let the perpendicular line from D to BC meet EF at G . Prove that the circles (ABC) and the circle centered at G with radius GD are tangent.



Proof. Let $P = DE \cap AC$, $Q = DF \cap AB$ and $T = DW \cap (ABC)$, then notice that,

Lemma 7.2 $AEQFPI$ is cyclic

Indeed, since P and Q are the reflections of D over CI and BI we know that,

$$\angle FPD = \angle FDP = \angle EQF$$

and,

$$\angle PFD = \frac{180 - 2\angle QDP}{2} = \angle A$$

consequently $APQEF$ is cyclic. By the Anchor Point Lemma we know that since as D moves PD and QD have constant directions and when D is the projection of I onto BC by Sharky-Devil we know that TD passes through W the midpoint of the arc BC . Thus TD always passes through W , but Anchor Point then tells us that the circle passes through a constant point lying on the isogonal conjugate of AW in $\angle CAB$ which implies that since I lies on the circle when D is the projection that I always lies on the circle. \square

Thus, since TD passes through W all we have to show to prove that (GD) is tangent to (ABC) is that T, G, O are collinear, where O is the circumcenter of $\triangle ABC$.

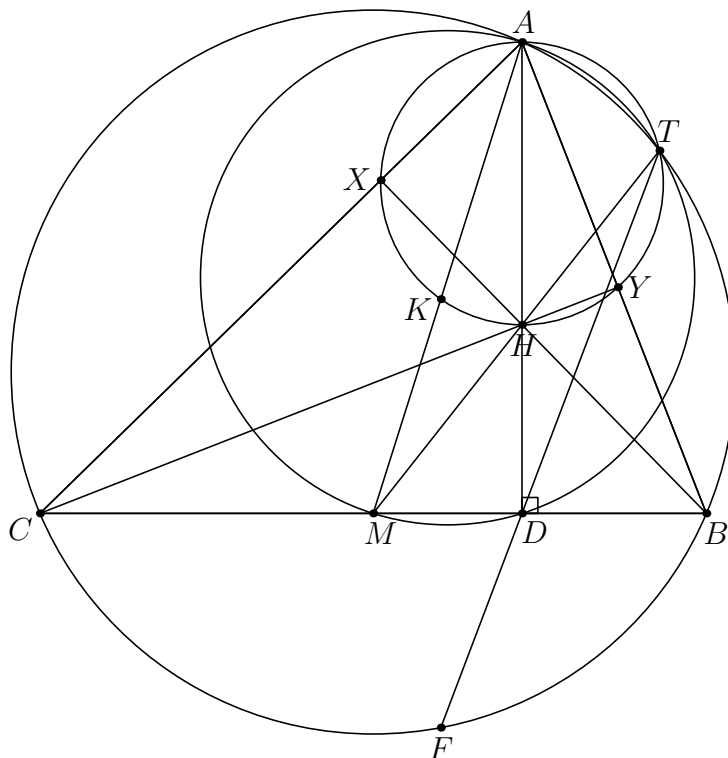
If D moves with degree 1 on BC then T moves with degree 2, similarly points F and E move with degree 1. Thus, the line EF moves with degree 2 and the line l (perpendicular from D to BC) moves with degree 1. Thus, G moves with degree 3. Thus, to show that T, G, O are collinear it suffices to consider $2 + 3 + 1 = 6$ cases.

Consider D being equal to B, C , the midpoint of BC , the projection of I onto BC (then one needs to prove that the reflection of D over G lies on (APQ) which is trivial by angle chase), $AI \cap BC$ and finally ∞_{BC} .

Consequently since T, G, O are collinear and T, D, W are collinear it must be that (GD) is tangent to (ABC) . ■

Problem 7.6 (APMO 2012 P4)

Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC , by M the midpoint of BC , and by H the orthocenter of ABC . Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH , and F be the point of intersection (other than E) of the line ED and the circle Γ . Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

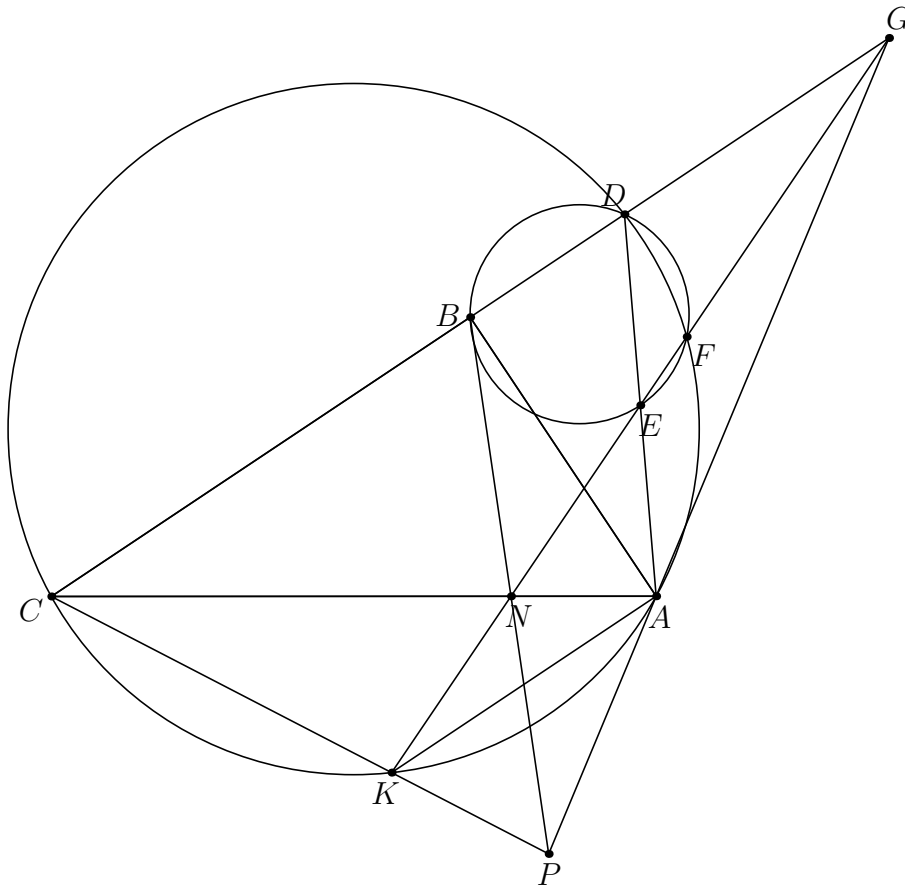


Proof. Notice, it is well known that T is the A -Simson point of ΔABC . Thus, if X and Y are the feet of the altitudes from B and C it must be that $AXHYT$ is cyclic. Notice, by **ELMO 2013 Shortlist G3** we obtain that since $AYDC$ and $AXDB$ are cyclic, consequently we know that (AXY) passes through a fixed point on the median, by Generalized Anchor Point we know that DT passes through the point F on (ABC) such that AF is the symmedian, thus $ABFC$ is harmonic and the problem statement follows. ■

Remark 7.3 The problem can also be solved by noticing that by Generalized Anchor Point since $ATMD$ is cyclic (since $\angle ATM = \angle ADM = 90^\circ$) it must be that TD passes through a point on (ABC) isogonal to AM w.r.t. $\angle CAB$.

Problem 7.7 (APMO 2022 P2)

Let ABC be a right triangle with $\angle B = 90^\circ$. Point D lies on the line CB such that B is between D and C . Let E be the midpoint of AD and let F be the second intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.



Proof. Let K be such that $CDAK$ is a trapezoid.

Let G be the reflection of C over B . Then let $P = \text{Anchor}_K^{\triangle ABC}(G)$ then is known that the Anchor Point conjugates of points on CD will lie on CK , thus $P \in CK$. However, it is also known that the Anchor Point transformation preserves cross-ratios which in the case of lines, implies the preservation of ratios thus since,

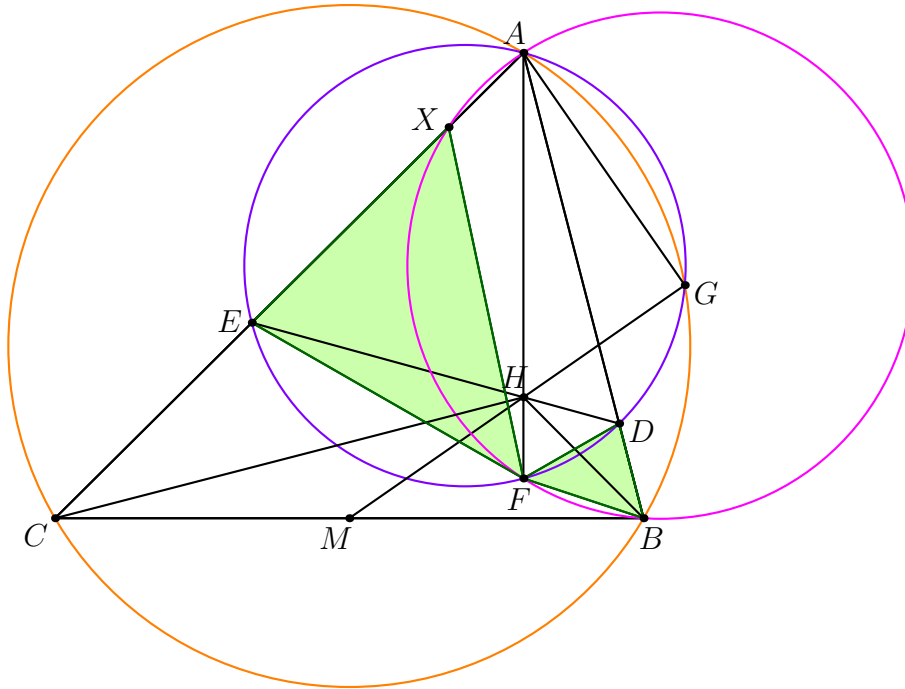
$$\begin{cases} \text{Anchor}_K^{\triangle ABC}(D) = K \\ \text{Anchor}_K^{\triangle ABC}(C) = C \end{cases}$$

thus we know that from the preservation of ratios that $DK \parallel GP$, however $AG \parallel DK$, thus G, A and P are collinear and K, E, G are collinear.

Then let $N = PB \cap AC$, trivially $N \in KG$, however now due to the definition of the Anchor Point conjugate $K \in NF$ which implies that $G \in EF$. ■

Problem 7.8 (IMO Shortlist 2005 G5)

Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC , and let M be the midpoint of the side BC . Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.



Proof. Let $X = (AFB) \cap AC$ and $F = AH \cap (AED)$. Since AF is the isogonal line of AA' (where A' is the antipode), then we know that the Evil Anchor Point conjugates of F with respect to A' are some infinity points.

Thus, by the *Evil Anchor Point characterization* theorem we must only show that $BX \parallel EM$, which is equivalent to E is the midpoint of CX . Notice,

Lemma 7.4 $\triangle DFB \sim \triangle EFX$

Indeed since $\angle EXF = \angle ABF$ and $\angle XEF = \angle FDB$. \square

Similarly,

Lemma 7.5 $\triangle CEH \sim \triangle BDH$

Indeed, since $\angle CEH = \angle HDB$ and $\angle ACH = \angle ABH$. \square

Consequently we know that,

$$\begin{cases} \frac{BD}{DF} = \frac{EX}{EF} \\ \frac{CE}{EH} = \frac{DB}{DH} \\ \frac{EH}{HD} = \frac{EF}{FD} \end{cases}$$

the desired result now trivially follows,

$$EX = \frac{BD}{DF} \cdot EF = BD \cdot \frac{EH}{HD} = EH \cdot \frac{CE}{EH} = CE$$

thus $CE = EX$. ■