Fourier Transform on $D_c^b(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}_l})$ and a proof of the last Weil Conjecture

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Fourier Transform

Let $\mathbb{F}_q$ denote the unique field with $q = p^r$ elements, and let l be prime number different than p. Let

$$\psi : \mathbb{F}_q \to \overline{\mathbb{Q}_l}$$

be a non-trivial additive character. (Question: How many of these are there?) This naturally induce characters

$$\psi : \mathbb{F}_{q^n} \xrightarrow{\mathrm{Tr}} \mathbb{F}_q \xrightarrow{\psi} \overline{\mathbb{Q}_l}$$

for all $n \in \mathbb{N}$.

Definition 1

Let $f : \mathbb{F}_{q^n} \to \overline{\mathbb{Q}_l}$ be any function. The *Fourier transform of* f is

$$\begin{aligned} T_\psi f : \mathbb{F}_{q^n} &\to \overline{\mathbb{Q}_l} \\ x &\mapsto \sum_{y \in \mathbb{F}_{q^n}} f(y)\psi(-xy). \end{aligned} \qquad \circ$$

The main theorems are just as for the usual real Fourier transform.

Theorem 2 (Plancherel Formula) For the n-norm

$$||f||_n := (f, f)_n = \sum_{x \in \mathbb{F}_{q^n}} f(x)\overline{f(x)},$$

we have $||T_\psi f||_n = q^{n/2}||f||_n$.

Theorem 3 (Fourier Inversion) $T_{\psi^{-1}} T_\psi f = q^n f$.

Exercise 1

Prove the theorems.

Given a sheaf $\mathcal{F}$ on a scheme X over $\mathbb{F}_q$ the geometric Frobenius automorphism

$$F \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q), a \mapsto a^{1/q}$$

acts naturally on the stalk $\mathcal{F}_{\overline{x}}$ for each geometric point $\overline{x} : \mathrm{Spec}\ \overline{\mathbb{F}_q} \to X$ of X.

Definition 4 (Sheaf-to-function Correspondence)

For $K \in D^b_c(X, \overline{\mathbb{Q}_l})$, we associate the function

$$\begin{aligned} f^K : X(\mathbb{F}_q) &\to \overline{\mathbb{Q}_l} \\ x &\mapsto \mathrm{Tr}(F, K_{\overline{x}}) := \sum_i (-1)^i \mathrm{Tr}(F, \mathcal{H}^i(K)_{\overline{x}}). \end{aligned} \qquad \circ$$

For $X = \mathbb{A}^1_{\mathbb{F}_q}$ we canonically identify $X(\mathbb{F}_q) \simeq \mathbb{F}_q$. Hence for $K \in D^b_c(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}_l})$ we can define the Fourier transform $T_\psi f^K$ of its associated function. The goal of these exercises will be to define the Fourier transform already on the level of the derived category. More precisely, given a non-trivial character $\psi : \mathbb{F}_q \to \overline{Q}_l$ we will construct a functor $T_\psi : D^b_c(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}_l}) \to D^b_c(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}_l})$ such that $f^{T_\psi K} = T_\psi f^K$. Furthermore, this Fourier transform should satisfy an appropriate analogue of the Fourier inversion theorem.

Exercises to be added!

Laumon's Proof of the Weil Conjecture

Let X be a smooth projective variety over $\mathbb{F}_q$ which is geometrically connected of dimension n. The fundamental question is to determine how many $\mathbb{F}_{q^n}$-points are in X. To study this, we collect this data in the *zeta function* of X, which is the formal power series

$$Z(X,t) := \exp(\sum_{n=1}^{\infty} \# X(\mathbb{F}_{q^n})\frac{t^n}{n}) \in \mathbb{Q}[[t]].$$

(Exercise: compute these for the point and $\mathbb{P}^1$.)
The Weil conjectures, now a theorem, is the following statement.

Theorem 5 (The Weil Conjectures)

1. *Rationality*: $Z(X,t)$ is a rational function.

2. *Functional Equation*: The zeta function satisfy the relation

$$Z(X, \frac{1}{q^n t}) = \pm q^{nE/2} t^E \cdot Z(X,t)$$

   where E is the self-intersection number of the diagonal $\Delta \subset X \times X$.

3. *Riemann hypothesis*: The rational function is of the form

$$Z(X,t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)},$$

   where each $P_i(t)$ satisfy

   (a) $P_0(t) = 1 - t$

   (b) $P_{2n}(t) = 1 - q^n t$

   (c) For $1 \leq i \leq 2n-1$

$$P_i(t) = \prod_j (1 - \alpha_{ij}t) \in \mathbb{Z}[t]$$

   where each α_{ij} is an algebraic integer and $|\alpha_{ij}| = q^{i/2}$ (where the norm is independent of embedding of $\mathbb{Z}[\alpha_{ij}] \hookrightarrow \mathbb{C}$!)

4. There should exist a cohomology theory analogous of analytical/topological cohomology, such that the rank of P_i is the ith Betti number of X.

In the 60's étale cohomology was developed and Grothendieck used it to prove part 1,2,4 and 3 except for (c). This was finally proven in 1974 by Deligne. In 1980 Deligne extended the framework of Weil's conjectures and proved a more general, yet more conceptual formulation of the Riemann hypothesis. In 1987 Laumon simplified a technical step in Deligne's proof using the Fourier transform discussed above. After an exercise session on the Fourier transform I will give a talk explaining Laumon's proof. Hopefully it will be somewhat in line with Magnus' talk.