Moduli Spaces

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Informal Definition 1. Let S be a set of mathematical objects. A moduli space is a surface, or higher dimensional manifold, whose points correspond naturally with the objects in S.

Example 1. *Consider the set of ways in which we can choose three (not necessarily distinct) points A, B, C on $\mathbb{R}^1$ (i.e. a line). The moduli space $\mathcal{M}_{\mathbb{R},3}$ of such configurations is isomorphic to $\mathbb{R}^3$: Indeed, to any point $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ we can associate the configuration of points A, B, C on $\mathbb{R}^1$ where the coordinate of A is x_1, the coordinate of B is x_2, and the coordinate of C is x_3.*

Problem 1. *Find the dimension of the following moduli spaces:*

(a) n (ordered) marked points on a line

(b) 3 (ordered) marked points in the plane

(c) 3 unordered marked points in the plane

(d) Triangles in $\mathbb{R}^2$ where one of the corners is $(0, 0)$.

(e) Triangles in the plane, up to translation and rotation

(f) Triangles in the plane, up to translation and rotation and scalation

(g) Lines in the plane (*)

(h) Lines in space (*)

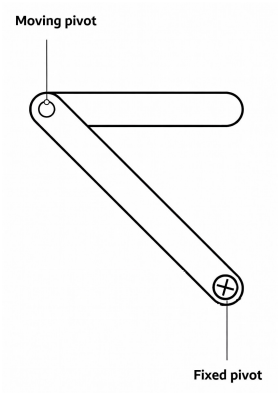
(i) Circles in the plane

(j) Parabolas in the plane (*)

(k) Ellipses in the plane (*)

(l) m-dimensional vector subspaces of an n-dimensional vector space (**)

Problem 2. *Consider the following linkage:*



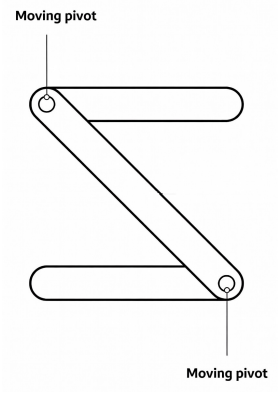
Let $\mathcal{M}_L$ denote the moduli space of the set of configurations the linkage can be in.

Let Ω be a fixed circle. We consider the set of ways in which we can choose two (not necessarily distinct) points A and B on the circle. Let $\mathcal{M}_{\Omega,2}$ denote the moduli space of configurations of two points on Ω.

Show that the surfaces $\mathcal{M}_{\Omega,2}$ and $\mathcal{M}_L$ are the same. Moreover, show that this surface can be obtained by taking a square and gluing together the opposite edges (i.e. it's a solid torus).

We denote the surface, corresponding to the moduli spaces from Problem 2, by $S^1 \times S^1$.

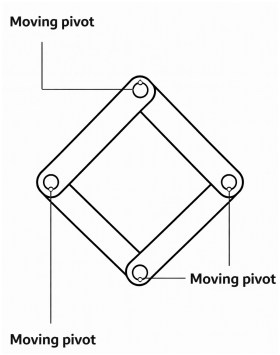
Problem 3. *Consider the following linkage:*



Let $\mathcal{M}_L$ denote the moduli space of the set of configurations the linkage can be in, this time up to translation and rotation.

Show that $\mathcal{M}_L$ is again isomorphic to the surface $S^1 \times S^1$ from Problem 2.

Problem 4. *Consider the following linkage:*



Let $\mathcal{M}_L$ denote the moduli space of the set of configurations the linkage can be in, up to translation and rotation.

Show that $\mathcal{M}_L$ is the union of three circles, where each pair of circles intersect in a point. Draw the moduli space, and explain which configurations of the linkage corresponds to the 3 "singular" points where the circles meet.

Problem 5. *Let Ω be a fixed circle. Consider the moduli space of pairs of (not necessarily distinct) unordered points on Ω. Describe this moduli space explicitly. (Hint available)*

References

[1] R. Pandharipande, *Moduli in Mathematics*, Hirzebruch Lecture at the Max Planck Institute for Mathematics and the University of Bonn, May 15, 2023. Available at: `https://people.math.ethz.ch/~rahul/2023-05-15_Hirzebruch_Lecture-Pandharipande.mp4`

[2] *Modulirum*, Mattekollo 2019, Lektionsmaterial, åk 9-gy2, Available at:

[3] Emily Mackevicius, *Configuration Spaces*. Available at: `https://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Mackevicius.pdf`