Knot Theory and applications to 3-4D Topology

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1 Introduction

We begin by looking at some fundamental questions to do with knots. Firstly, what is a knot and how do we define whether two knots are equivalent. Secondly, how do we tell them apart in practice?

All spaces here are topological (and often will have more structure but this will be stated) and maps are continuous unless stated otherwise.

Definition 1. *(Knot) A knot is an embedding $S^1 \to S^3$. We can specify the knot to be topological if the embedding is just continuous and smooth if the embedding is smooth.*

Most often we will call a knot K the image of an embedding of S^1 in S^3 the latter of which we view as $\mathbb{R}^3 \cup \{\infty\}$. Higher dimensional knots are just embeddings of S^k into S^{n+k}. So to study knots we need to understand a bit more about embeddings.

Definition 2. *Two embeddings $f, g : X \to Y$ are equivalent if there is a homeomorphism $h : Y \to Y$ s.t. $f \circ h = g$*

Check 1. *Does the definition above change if we require instead $h \circ f = g$, what about $g \circ h = f$*

We will want to understand which embeddings can be deformed in the ambient space into one another. Our equivalence relation will be isotopy. To help with intuition and to fit with what is to come, we will restrict ourselves to $X = S^1$ and $Y = S^3$ even though a lot of the concepts below are generalisable.

Definition 3. *(Isotopy) Two knots $K, L \subset S^3$ are isotopic if there is a family of embeddings $f_t : S^1 \to S^3$ s.t. $t \in [0, 1]$ and $f_0(S^1) = K, f_1(S^1) = L$ and the map $F : S^1 \times [0, 1] \to S^3$ defined by $(x, t) \mapsto f_t(x)$ is continuous in the product topology.*

Once again we may choose the embeddings to be either topological or smooth. However it will turn out that only one definition captures the behaviour of knots that we base on real life.

Exercise 1. *Show that under one of the definitions of smooth or topological isotopy the unknot is equivalent to the trefoil knot.*

[insert trefoil]

In fact the question of whether any knot is isotopic to the unknot is currently a big open problem. The current contender is 'wild' knot called the Bing Sling. Before talking more about 'wild' knots. We describe a correct equivalence relation.

Definition 4. *(Ambient isotopy) Two knots K_1, K_2 are ambient isotopic if they are isotopic and there is a map $G : S^3 \times I \to S^3$ where $G_0 = id_{S^3}$ and $f_0 \circ G_1 = f_1$*

Exercise 2. *Given an ambient isotopy between two knots define an isotopy between the same knots.*

We will need to restrict ourselves to studying a certain class of knots to rule out some pathological constructions like the Bing Sling. Although the theory of these knots is very interesting and much is still open there we will not study it here for the moment.

Definition 5. *A knot is called PL (or piecewise linear) if it's image in $\mathbb{R}^3$ is a countable union of segments and points (according to the affine structure of $\mathbb{R}^3$.*

Definition 6. *(Equivalence of embeddings) Two embeddings $f, g : X \to Y$ (X, Y topological spaces or manifolds if we require more structure) are equivalent if there is a homeomorphism $h : Y \to Y$ such that $h \circ f = g$.*

Definition 7. *(tame and wild) A knot K is tame if it is equivalent to a PL knot. Conversely if a knot is not equivalent to any PL knot it is called wild. A wild knot may sometimes be PL apart from a subset of points $W \subset K$, points in the set are referred to as wild.*

Examples of wild knots are non-trivial to construct. The first wild knot was first discovered by Alexander and it fails to be PL at a Cantor set of points. The main idea in the proof was showing that the complement of the wild knot is not simply connected. Simpler examples followed, constructed by Fox and Artin. They constructed knots that are wild in a finite number of points. It only suffices to have one point where the embedding is not PL to create a wild knot (although this only holds in dimension three!), however the compliment of such a knot will have a simply connected component so showing this knot is wild required a little more work.

[insert wild knots]

1.1 Projections, Riedmeister moves and Invariants

Often we will keep a two-dimensional picture when we will deal with knots. The correctness of this notation follows from the following exercise.

Exercise 3. *Let $K \subset S^3 = \mathbb{R}^3 \cup \infty$ be a knot. There is a projection $\pi(K)$ onto a plane such that everywhere apart from a finite number of points $P \subset \pi(K)$ the projection is an embedding and each point $p \in P$ has exactly two pre-images.*

Such a projection is called a knot diagram. It is a remarkable result by Reidmester that we can encode equivalence of knots purely by a sequence of three moves.

Reidmeister moves...

Exercise 4. *Simplify the following diagram of the unknot.*

Exercise 5. *Show that every knot has a bridge representation.*

A knot crossing is a double point in the projection with the added information of which strand is above the other. We can perform a crossing change which switches the above/below order of the strands.

Given a knot K, what is the minimal number of crossing changes needed to make it into the unknot.

It turns out this is a very difficult question in general but it leads to some nice measures of complexity of a knot.

Definition 8. *The unknotting number of a knot K is the minimal number of crossing changes needed to obtain the unkot. This is denoted $u(K)$*

Theorem 1. *The unkotting number detects the unkot i.e. $u(K) = 0$ iff K is the unknot.*

Exercise 6. *What is the unknotting number of the trefoil?*

The main problem is that $u(K)$ is extremely difficult to compute in general. There are algorithms that get you upper bounds on the $u(K)$ however we often cannot show that these are minimal. Can you find such an algorithm?

There are some very simple conjectures that are amazingly still open in this field.

Conjecture 1. *Let K_1, K_2 be knots then $u(K_1K_2) = u(K_1) + u(K_2)$*

Exercise 7. *Prove one side this conjecture (i.e. write down an prove an inequality)*

Exercise 8. *Show that the Whitehead double of any knot has unknotting number* 1.

One can also associate some information to the crossings of a knot. Below a figure illustrates the difference between a positive and negative crossing.

[crossings]

Using the crossing information one can construct an important knot invariant: the Jones polynomial. First we define the Kauffman bracket.

1.2 Siefert surfaces and Genus

Most of the very interestinng theory comes from not just studying knots and their equivalence classes by themselves but constructing more interesting spaces from a knot. For example one can construct any three-manifold starting from a knot (we will come back to this later). A more basic construction is a Siefert surface.

Definition 9. *(Seifert Surface) Given a knot $K \subset S^3$ a Siefert surface is a surface (a 2 dimensional manifold with boundary) $S \subset S^3$ whose boundary is K.*

Proposition 1. *Every knot has a Siefert Surface*

This definition is not unique, a knot can have infinitely many Siefert surfaces, since if we find one then we can find many more by just 'drilling' a cylinder through the surface toi increase its genus (for those who are familiar with handle geometry we are adding a 1-handle to $S \setminus \partial S$. This motivates the following.

Definition 10. *(Seifert genus) The Siefert genus of a knot is the minimal genus of all Siefert surfaces for that knot. denoted $g_3(K)$.*

The Siefert genus is important because for one it detects the unknot. That is if $g_3(K) = 0$ the K is the unknot. Furthermore it is also a knot invariant.

Exercise 9. *Check that $g_3(K)$ remains unchanged under Riedmester moves.*

It is a remarkable coincidence that for quite a lot of low crossing knots the Siefert genus and the genus of the surface obtained using Siefert's algorithm are the same. However in general just like the unknotting number finding the Siefert genus is not a simple task, however there are algorithms in place as well as fancy theory (Knot Heegard Floer homology for example).

Another useful result concerning the Siefert genus is that t behaves well under connected sum.

Theorem 2. *Let K_1, K_2 be knots then $g_3(K_1 \# K_2) = g_3(K_1) + g_3(K_2)$*

One can look at Siefert surfaces in one dimension higher. Picture $S^3 = \partial B^4 \subset B^4$, one can then ask what is the minimal genus of a surface in B^4 with boundary the knot $K \subset S^3$. This is denoted $g_4(K)$ and is reffered to as the slice genus of the knot. A knot is called slice if $g_4(K) = 0$. Again one can ask whether a knot is smooth slice or topologically slice depending on whether it bounds a smooth or topological disc. This distinction is extremely important in four-dimensional topology. The existence of topologically slice knots that are not smooth slice imply the existence of exotic structure of four-manifolds. To bring into context how important such a result is, we note that the existence of an exotic structure on S^4 is the last open problem in the Poincare conjecture series and is referred to as the smooth 4-d Poincare conjecture.

Exercise 10. *Relate the unknotting number, Siefert and slice genus by proving the following properties: $g_3(K) \leq u(K)$, $g_3(K) \leq g_4(K)$, $g_4(K \# m(K)) = 0$*

1.3 Heegard Decompositions

When studying three-manifolds one of the first problems that arises that is different from the previous dimension, two, is how to talk about a three-manifold. For surfaces we have the standard picture of a sphere with a number of handles attached to it. Obtaining such a combinatorial description of a three manifold is more comlicated. Of course we can consider 3-manidolds to have a simplicial or CW structure but often this gives limited information an we cannot answer classification problems.

Let us start with an example. Consider a ball D^3 then $^3 = S^2$. This boundary S^2 can carry a cell structure and we can identify different cells together to form a new space. When is this quotient a 3-manifold?

Theorem 3. *Let X be obtained by the pairwise identification of 2-cells on the boundary S^2 of D^3 as descirbed above. Then X is a cosed 3-manifolds iff its Euler Characteristic is 0.*

-example Lens spaces

A very important result in 3-dimensional topology is due to Heegard. It essentially classifies 3-manifolds.

Construction 1. *(Heegard Diagrams)*

A Genus g Heegard splitting of a connected ,closed oriented three-manifold Y is a decomposition of $Y = U_\alpha \cup_\Sigma U_\beta$, where Σ is an oriented, connected, closed 2-manifold with genus g and U_α and U_β are handlebodies with $\partial U_\alpha = \Sigma = -\partial U_\beta$. It is knwon that every closed, oriented three-manifold admits Heegard spitting (but it is not unique).

A handlebody U can be described by attaching g two-handle along g disjoint simple closed curves $\{\gamma_1, \gamma_2, \ldots, \gamma_g\}$ which are linearly independent in $H_1(\Sigma, \mathbb{Z})$, and then one three-handle. The curves γ_i are called attaching circles for U. Since the three-handle is unique, U is determined by the attaching circles. However the reverse is not true, attaching circles are not uniquely determined by U, for example they can be moved by isotopies.

Thus one can think of a genus g splitting of a closed three-manifold $Y = U_\alpha \cup_\Sigma U_\beta$ as specified by a genus g surface Σ and a pair of attaching circles $\alpha_{\alpha_1,\ldots,\alpha_g}$ and $\beta = \{\beta_1, \ldots, \beta_g\}$ whih are attaching circles for the U_α and U_β handlebodies respectively. The triple (Σ, α, β) is called a Heegard diagram.

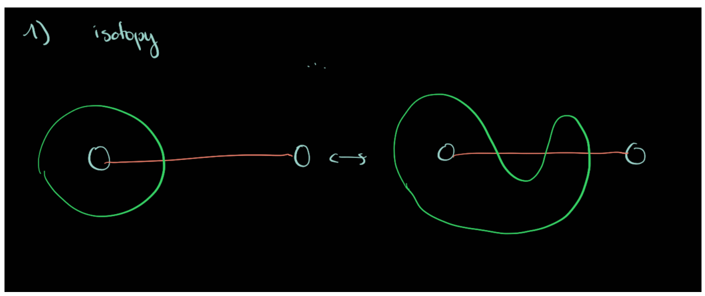
Exercise 11. *List all the spaces which have a Heegard decomposition of genus one.*

Note that Heegard diagrams have a Morse-theoretic interpretation s follows. Let $f : Y \to [0,3]$ be a self indexing Morse function on Y with one minimum and one maximum, then f induces as Heegard decomposition with surface $\Sigma = f^{-1}\frac{3}{2}$, $U_\alpha = f^{-1}[0,\frac{3}{2}]$, $U_\beta = f^{-1}[\frac{3}{2},3]$. The attaching circles are the intersection of Σ with the acending and descending manifolds for the index one and two critical points respectively (after some choice of Riemmanian metric on Y). Such a Morse function is called compactible with the Heegard diagram.

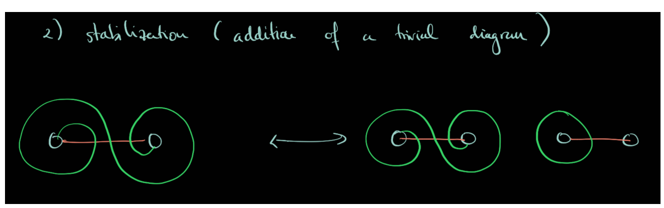
In what follows, we adopt the convention that Heegard diagrams are drawn on the plane. To see this we view a surface of genus g as a sphere S^2 with $2g$ marked discs (each pair of discs specifies the attachment of a one-handle). Then the sphere is viewed as the plane union infnity.

Whilst different Heegard diagrams can specify the same manifold, one can introduce an equivalence relation on them using three properties. Let (Σ,α,β) and (Σ',α',β') be a pair of Heegard diagrams.

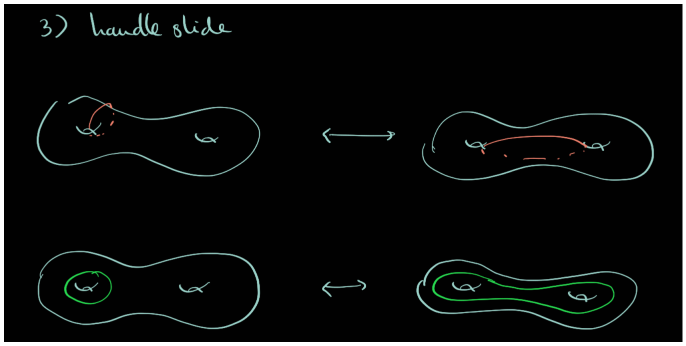
Definition 11. *(Heegard Moves) 1) We say two Heegard diagrams are isotopic if $\Sigma = \Sigma'$ and there are two families α_t, β_t of g-tuples of curves, moving by iotopies so that for each t, both the α_t and β_t are g-tuples of smoothly embedded, pairwise disjoint curves*



2) We say (Σ',α',β') is obtained from (Σ,α,β) by stabilization if $\Sigma' \cong \Sigma \# E$ and $\alpha' = \{\alpha_1,\dots,\alpha_g,\alpha_{g+1}\}$, $\beta' = \{\beta_1,\dots,\beta_g,\beta_{g+1}\}$ where E is a torus $S^1 \times S^1$, and α_{g+1},β_{g+1} are a pair of curves in E which meet transversally in a single point. Conversely, in this case we say that (Σ,α,β) is obtained from (Σ',α',β') by destabilization.



3) We say that (Σ',α',β') is obtained from (Σ,α,β) by handleslides if $\Sigma' = \Sigma$ and for $\gamma \in \{\alpha,\beta\}$ $\gamma' = \{\gamma_1',\gamma_2,\dots,\gamma_g\}$ where $\gamma_1',\gamma_1,\gamma_k$ $k \neq 1$ bound a pair of pants. (In this case γ_1' is the result of sliding γ_1 over γ_k.)



Then we have the following result.

Proposition 2. *Any two Heegard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ which specify the same three manifold are diffeomorphic after a finite sequence of Heegard moves.*

Exercise 12. *By choosing a certain number of points $z_i \in \Sigma - \alpha \cup \beta$ for $i = 1, \ldots, n$ we can define a multi-pointed Heegard diagram as $(\Sigma, \alpha, \beta, z)$ where (Σ, α, β) is a Heegard diagram in the previous sense and $z = (z_1, \ldots, z_n)$.*

Then then a pointed Heegard move is a Heegard move that leaves the basepoints fixed. For example one must isotope the curves without touching the basepoints.

Show a sequence of Pointed Heegard moves relating these two Heegard diagrams.

1.4 Problems

Problem 1. *Show that if $K \subset S^3$ is a non-trivial knot then there is a straight line that meets K in four points.*

Problem 2. *Show that the number of three colourings of a knot is a knot invariant (i.e. it remains unchanged under Riedmeister moves). What about the number of n-colourings?*

Problem 3. *Show that a smooth knot is ambiently equivalent to a PL knot, and vise versa i.e. if $K \subset S^3$ is a PL knot then there is a PL knot $L \subset S^3$ and a homeomorphism $h : S^3 \to S^3$ s.t. $h(K) = L$.*

Problem 4. *The algebraic surface $f(z, w) = z^a + w^b = 0$ with $z, w \in \mathbb{C}$ and $a, b \in \mathbb{Z}$, $a, b \geq 2$ intersects S^3 (viewed as a neighbourhood of the origin in $\mathbb{C}^2$ in a torus knot $T_{a,b}$.*

Problem 5. *The Klee-Trick (see Daverman-Venema for details) is a trick by which one can show that an interval embedded in $\mathbb{R}^n$ when viewed as a subset of $\mathbb{R}^{n+1}$ is equivalent to the standard embedding of an interval in $\mathbb{R}^{n+1}$.*

Show that any PL embedding of an interval in $\mathbb{R}^n$ where $n > 3$ is equivalent to the standard one. (Hint: Find a projection onto $\mathbb{R}^3$ and use the Klee Trick)

Problem 6. *The linking number of a knot*

Problem 7. *Using the fact that $\pi_1(M^3)$ have balanced presentations (a presentation with as many relations as generators) show that unlike the two-dimensional case $\mathbb{Z}_4$ cannot be the fundamental group of some three-manifold. Find another example.*

1.5 Potential Projects or Talks

Below is a list of topics on which students can give a talk on, or do a project in. The aim of a project is to understand a topic very well and to convey this understanding into written form. Depending on the project the final result may be publishable. To do a project please email me in advance.

Topic 1. *Show that the study of knots* $S^1 \to S^2$ *is trivial. In other words any embedding* $S^1 \to S^2$ *can be isotoped into the standard one. Consider the different cases when the embedding is smooth or just topological. Now generalise to codimension* 1 *knots* $S^{n-1} \to S^n$.

Topic 2. *(Rolfsen) Investigate whether any knot isotopic to the unknot? Start by showing that any knot that is locally tame at one point is tame. Consider the Alexander Horned sphere* H *as depicted below and the knot* K_H *that arises by intersecting* H *with a plane going through the wild set. Is* K_H *tame?*

Topic 3. *Read and give a talk about a wild construction. For example Fox and Artin's paper titled wild 3 cells, or some more elaborate constructions such as Bing's hooked rug.*

Topic 4. *Show that every embedded sphere in* $\mathbb{R}^3$ *bounds a 3-ball. Investigate* $\pi_2(S^1 \times S^2)$. *Are there any other* 3*-manifolds with a similar* π_2*?*