Exercises on Complex Analysis and Special Functions

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1 Gamma Function

1. Prove that
$$\Gamma(z) = \frac{1}{z}\prod_{j=1}^{\infty}(1+\frac{z}{j})^{-1}(1+\frac{1}{j})^{z}.$$

2. How many solutions are there to $\Gamma(z) = 0$?

3. Compute
$$\int_0^1 \sqrt{-\log(x)}\,dx,$$
what is $\int_0^1 (-log(x))^n\,dx$?

4. It was shown that the Gamma function can be analytic continued by inductively defining $\overline{\Gamma(z)}$ as $\frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}$ depending on the value of $\Re(z)$. What are the residues (and poles) of the Gamma function? Remark that this can be done for any holomorphic function $f(z)$ (that is entire on the half plane) and satisfies the functional equation $f(z+1) = zf(z)$.

5. Let $f(z)$ be a holomorphic function on the upper half plane i.e the set $A = \{z : 0 < \Re(z)\}$, also let $f(1) = 0$, show that this function has an analytic continuation that is *entire*.

6. (*) Now consider a function $g(z)$ that is holomorphic on the upper half plane, satisfies the Gamma functional equation and is also bounded on the strip $B = \{z : 1 \leqslant \Re(z) < 2\}$, now by considering $f(z) = g(z) - g(1)\Gamma(z)$ and $f(z)f(1-z)$ show that $f(z) = 0$. This result is called *Wielandt's theorem.* (Hint: Use Liouvilles theorem)

7. Show that
$$B(z,z) = 2^{1-2z}B(1/2,z),$$
where $B(z,w)$ is the beta function.

2 Inversion Theorems

Recall that the *Fourier transformation* of a piecewise-smooth absolutely integrable function is given by

$$(\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} f(x)\, dx,$$

and accordingly it's inverse transform is given by

$$(\mathcal{F}^{-1}g)(x) = \int_{-\infty}^{\infty} e^{2\pi i x \xi} g(\xi)\, d\xi.$$

1. Define the *Mellin transform* of a piecewise continuous function $f : \mathbb{R}^+ \to \mathbb{C}$ by

   $$(\mathcal{M}f)(s) = \int_0^{\infty} x^{s-1} f(x)\, dx$$

   and it's inversion by

   $$(\mathcal{M}^{-1}\phi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} x^{-s} \phi(s)\, ds,$$

   where $\phi(s)$ is an analytic function on the strip $a < \Re(s) < b$ that tends to 0 (uniformly) as $\Im(s) \to \pm\infty$, and c is any real number on the interval (a, b) and $x \in \mathbb{R}^+$. Notice that we are integrating over the line parallel to the imaginary axis that passes through the point c. Show (with the help of *Fourier inversion*) that

   $$(\mathcal{M}^{-1}\mathcal{M}f)(x) = f(x).$$

   (Hint: Consider the Fourier (and inverse Fourier) transform of $\widetilde{f} = f \circ e^x$, notice that e^x is a bijection here.)

2. Compute

   $$\int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{s-1}\, ds,$$

   ,

   $$\mathcal{M}(\frac{x}{x+1}), \mathcal{M}(\frac{1}{(x+1)^{m+s}}), \mathcal{M}(f(ax)), \mathcal{M}(x^a f(x))$$

3. (**) Prove Ramanujan's Master theorem A,

   $$\mathcal{M}\left(\phi(0) - \phi(1)x + \phi(2)x^2 - \phi(3)x^3 \ldots\right) = \frac{\pi}{\sin(\pi s)} \phi(-s),$$

   where $\phi(s)$ is an analytic function such that $|\phi(s)| < e^{P\Re(s) + E\Im(s)}$, and $0 < x < e^{-P}$.

   (Hint: Calculate the integral $\frac{1}{2\pi i}\int_C \frac{\pi}{\sin(\pi s)} \phi(-s) x^{-s}\, ds$ with the help of the residue theorem. Where C is a certain half-circle, Jordan's Lemma might help here too.)

4. Prove Ramanujans Master Theorem B,

$$\mathcal{M}(\phi(0) - \phi(1)\frac{x}{1!} + \phi(2)\frac{x^2}{2!} - \phi(3)\frac{x^3}{3!} \ldots) = \Gamma(s)\phi(-s).$$

5. Compute

$$\int_0^\infty \frac{x^{s-1}}{1+x}\,dx, \int_0^\infty W_0(x)x^{s-1}\,dx, \int_0^\infty \frac{x^s}{e^x+1}\,dx, \int_0^\infty \frac{x^s}{e^x-1}\,dx$$

using the *Master theorem*. ($W_0(x)$ denotes the Lambert W function with an expansion around 0)

6. Recall that the expansion of $\frac{x}{e^x-1}$ is given by $\sum_{k=0} \frac{B_k}{k!}x^k$ where B_k denotes the Bernuolli numbers. Show that,

$$\zeta(1-n) = (-1)^{n+1}\frac{B_n}{n},$$

using the *Master theorem*. (Be careful with the signs and only keep to whole numbers, also notice what happens to the Gamma function.)

7. (***) Prove that

$$\sum_{n>0\,odd} \frac{n}{e^{n\pi}+1} = \frac{1}{24},$$

(Hint:Use the function $f(x) = \frac{\pi x}{e^{\pi x}+1}$, together with its *inverse Mellin Transform* and try to sum up the odd values.)

8. Using a similiar method, evaluate

$$\sum_{n>0\,n\,odd} \frac{1}{e^{n\pi}-1}, \sum_{n>0} \frac{1}{e^{n\pi}+1}, \sum_{n>0\,n\,odd} \frac{n^2}{e^{n\pi}-1}$$

3 The Riemann Zeta function

1. Show that the Zeta function has no nontrivial zeroes outside the critical strip $\{z : 0 \leqslant \Re(z) \leqslant 1\}$

   (Hint: Use the Euler product formula together with the functional equation. If you want more of challenge, then show that the Zeta function has no (non-trivial) zeroes on the lines $\Re(z) = 0$ and $\Re(z) = 1$.)

2. Using the functional equation, compute $\zeta(0)$ and $\zeta(-1)$.

3. Show that an analytic continuation for zeta function on the upper half plane can be given by

$$\frac{1}{1-2^{1-s}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k^s},$$

the sum $\eta(s) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k^s}$ is of major importance in of itself and is called the *Dirchlet Eta function*.

3.1 Another Derivation of the functional equation

Let $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ denote the *complete zeta function*. Let $\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{n^2\pi s}$ denote the Jacobi theta function.

4. *Poisson summation formula*, Prove that for sufficiently nice function (i.e C^2 and absolutely integrable) show that,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

(Hint:Look at the function, $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ and try to find a corresponding Fourier transform, *what is the period of this function?*)

5. Prove the Theta functional equational

$$\vartheta(s) = \frac{1}{\sqrt{t}}\vartheta(1/t).$$

(Hint:Try to use the Poisson Formula)

6. *Part 1*, Begin with proving that

$$\pi^{-s/2}\Gamma(s/2)n^{-s} = \mathcal{M}(e^{-n^2\pi t})\{s/2\},$$

and therefore

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \mathcal{M}\left(\frac{\vartheta(t)-1}{2}\right)\{s/2\}.$$

Let now $\psi(t) = \frac{\vartheta(t)-1}{2}$.

7. *Part 2*, Show that

$$\psi(t) = \frac{t^{-1/2}(2\psi(1/t)+1)-1}{2},$$

and therefore that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \mathcal{M}\left(\psi(t)\right)\{s/2\} = \frac{1}{s(s-1)} + \int_1^{\infty} \psi(t)(t^{\frac{s}{2}-1} - t^{-\frac{1}{2}+\frac{s}{2}})\,dt$$

8. *Part 3*, Notice that the expression to the right of the above equation is invariant under the transformation $s \mapsto 1-s$, and conclude that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \leftrightarrow \Lambda(s) = \Lambda(1-s),$$

using various formulas throughout the exercises (in particular Euler's and Legendre's reflection formulas), show that

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

This also gives us the analytic continuation of the Zeta function throughout the complex plane.

9. Prove that the trivial zeroes of the Zeta function are the negative even integers.