Introduction to Birational Geometry

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1. Blowing Up the Plane

Consider $p = (0,0) \in \mathbb{C}^2$. We let $\mathbb{P}^1$ denote the set of lines through p. Such a line is defined by an equation on the form $ux + vy = 0$ for a pair of complex numbers $(u,v) \neq p$. The pairs (u,v) and (u',v') determine the same line if and only if there exists a non-zero complex number α such that $u' = \alpha u$ and $v' = \alpha v$. If this is so, we say that $(u,v) \sim (u',v')$. It follows that

$$\mathbb{P}^1 \cong (\mathbb{C}^2 \setminus p)/\sim .$$

Let $\mathbb{C}^2 \times \mathbb{P}^1$ denote the set of pairs (x,ℓ) with $x \in \mathbb{C}^2$ and $\ell \in \mathbb{P}^1$.

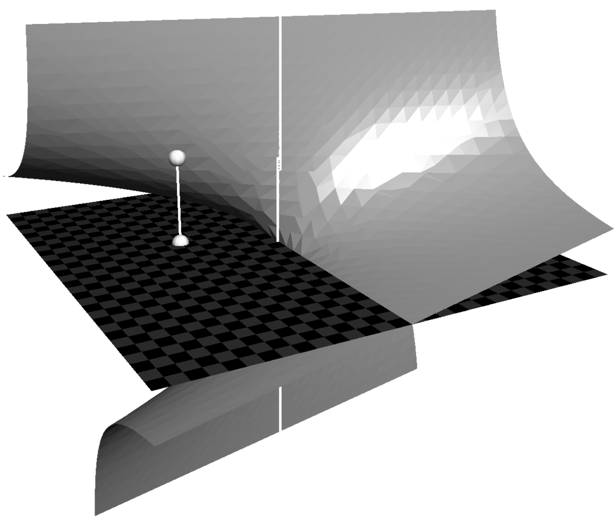
Definition 1.1. $Bl_p(\mathbb{C}^2) \subset \mathbb{C}^2 \times \mathbb{P}^1$ denotes the set of pairs of points and lines such that the point lie on the line.

$$Bl_p(\mathbb{C}^2) = \{(x,\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 : x \in \ell\}$$

While we have coordinates (x,y) on $\mathbb{C}^2$, we have *homogenous coordinates* $(u:v)$ on $\mathbb{P}^1$. A point in $\mathbb{C}^2 \times \mathbb{P}^1$ is given by coordinates $(x,y) \times (u,v)$.

Exercise 1.1. $Bl_p(\mathbb{C}^2) \subset \mathbb{C}^2 \times \mathbb{P}^1$ is defined by the equation $xv - yu = 0$.

Let U_1 define the subset of $Bl_p(\mathbb{C}^2)$ with $u \neq 0$, and let U_2 define the subset of $Bl_p(\mathbb{C}^2)$ with $v \neq 0$. Then $Bl_p(\mathbb{C}^2) = U_1 \cup U_2$. If $(x,y) \times (u:v) \in U_1$, we have $(u:v) \sim (1:v/u)$. A point in U_1 is determined by a tripple $(x,y,v/u)$ in $\mathbb{C}^3$ satisfying $x \cdot v/u - y = 0$. So U_1 is isomorphic to the quadric surface in $\mathbb{C}^3$ defined by the equation $x \cdot v/u - y = 0$. The map $\pi : U_1 \to \mathbb{C}^2$ is just the orthogonal projection onto the $v/u = 0$ plane.



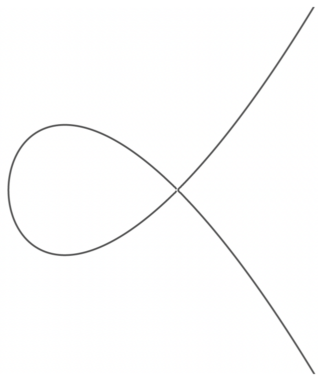
Let $\pi : Bl_p(\mathbb{C}^2) \to \mathbb{C}^2$ denote the projection, that maps a pair (x,ℓ) to x. $\pi^{-1}(p) \cong \mathbb{P}^1$ is called the exceptional divisor and is denoted by E.

Exercise 1.2. $\pi : Bl_p(\mathbb{C}^2) \setminus E \to \mathbb{C}^2 \setminus p$ is an isomorphism.

Let D be a curve in $\mathbb{C}^2$ containing p. Then $\pi^{-1}(D)$ will consists of two intersecting components; the exceptional divisor E and a curve D' called the strict transform of D. We can also define the strict transform D' as the closure of $\pi^{-1}(D)$.

Exercise 1.3. Let $l_1, l_2 \subset \mathbb{C}^2$ denote two lines intersecting in p. Prove that l_1', l_2' do not intersect.

Exercise 1.4. Consider the nodal cubic curve C defined by $y^2 = x^3 + x^2$, with a singular double-point in p. Find a parametrization of C, and use that to find a parametrization of the strict



transform C' in U_1 and in U_2. Prove that C has no double points.

Exercise 1.5. Prove that the strict transform of the cuspidal cubic $y^2 = x^3$ is tangent to the exceptional divisor.

2. Problems

2.1. Prove that the blow up of $\mathbb{P}^2$ in two points is isomorphic to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point.

2.2. Let $p = (0,0) \in \mathbb{P}^2$. Consider the map $f : \mathbb{P}^2 \setminus p \to \mathbb{P}^4$ defined by

$$(x,y) \mapsto (x_0 x_1 : x_0 x : 2 : x_1 x_2 : x_2^2).$$

Prove that the map extends to give a closed immersion $Bl_p(\mathbb{P}^2) \to \mathbb{P}^4$, prove that straight lines through p are mapped to straight lines that do not intersect. Prove that the image is a surface of degree 3 (that is, it has 3 intersection points with a general 2-dimensional plane).

Definition 2.1. A rational map $X \to Y$ is a map from an open subset of X to Y.

2.3. A smooth quadric $Q \in \mathbb{P}^3$ is defined by an equation of degree 2, and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let $x \in Q$. Then the projection from Q to a plane through x gives a rational map $\psi : Q \to \mathbb{P}^2$. On what open subset of Q is this map defined, and what is the image of this map? Decompose $\psi = \phi \circ \varphi^{-1}$ for morphisms

$$Q \xleftarrow{\varphi^{-1}} X \xrightarrow{\phi} H,$$

where ϕ, φ are blowups, perhaps at several points.

3. Minimal Resolutions of Singularities

Consider the cone X in $\mathbb{C}^3$ defined by $xy - z^2 = 0$, and let $p = (0,0,0)$.

3.1. Describe the blow up $Bl_p(X)$ of X in p. Is it smooth?

3.2. Let C be a curve with one singular point p. Can $Bl_p(C)$ still be singular?

Definition 3.1. For a singular variety X with singular locus $Sing(X)$, a minimal resolution of singularities is a smooth variety Y with a morphism $f: Y \to X$, such that for all other morphisms $f': Y' \to X$ with Y' smooth, we can write $f' = f \circ g$ for some morphism $g: Y' \to Y$.

3.3. Let $\pi \subset \mathbb{C}^4$ be a 2-dimensional plane. Let X denote the set of 2-dimensional planes in $\mathbb{C}^4$ that intersect π in dimension at least 1. Describe $Sing(X)$. Is $Bl_{Sing(X)}(X)$ smooth? Is it a minimal resolution?

3.4. Find a singular variety that does not have a minimal resolution of its singularities.

4. Mixed

4.1. Prove that $\mathbb{P}_{\mathbb{P}^1}(\oplus(-1)) \cong Bl_p(\mathbb{P}^2)$.

4.2. Prove that the automorphism group of the blow up of $\mathbb{P}^2$ in 4 points (no three colinear) is the permutation group of 5 elements.