Problem set for Hodge Theory

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1. Basic ones

Exercise 1.1 (*More general and not related to classes*)**.** Holomorphic maps preserve the orientation of a complex manifold and complex manifolds posesses an orientation.

Exercise 1.2. Recall that we have defined complex differential maps as $\partial:\Omega^{p,q}\to\Omega^{p+1,q}$ and $\overline{\partial}:\Omega^{p,q}\to\Omega^{p,q+1}$. Show that we have

$$\overline{\partial}^2=\partial\overline{\partial}+\overline{\partial}\partial=\partial^2=0.$$

Exercise 1.3 (*Riemann surfaces*)**.** Show that any Hermitian metric on Riemann surface is Kähler.

Exercise 1.4 (*Fubini-Study Metric on* $\mathbb{CP}^n$)**.** Let us define it in the following way. Let π be a canonical projection $\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{CP}^n$. For a small enough open set U pick some holomorphic section $F:U\to\mathbb{C}^{n+1}\setminus\{0\}$ of π. For any $p\in U$ consider:

$$||F(p)||^2=\sum_{i=0}^{n}F_i(p)\overline{F_i(p)}\neq 0$$

Then we can pick U small enough such that $\log||F||^2$ is defined. Now set:

$$\omega_F=\frac{i}{2\pi}\partial\overline{\partial}\log||F||^2.$$

The main goal is to show that this form define properly Hermitian metric.

- Show that this definition doesn't depend on the choice of holomorphic section F.
- Show that ω is in fact $(1,1)$-form.
- Show that we can glue forms ω_F to the global one.
- Check that this positively defined form. (One can show that it is enough to check around one point, say $(1:0:\cdots:0)$).
- Show that this form is closed, i.e. this is a Kähler manifold.

Exercise 1.5. Suppose that in the basis e_i the quadratic form is given by the matrix $g=g_{ij}$, and write the inverse matrix $g^{-1}=(g^{ij})$. Prove that for a 1-form $\alpha=\alpha_i e^i$ one has

$$*\alpha=(-1)^{i-1}g^{ij}\alpha_j e^1\wedge\cdots\wedge e^{i-1}\wedge e^{i+1}\wedge\dots e^n.$$

Exercise 1.6. Proof that $*^2=(-1)^{p(n-p)}$ on Ω^p_X.

2. A little bit more advanced

Exercise 2.1. Show that if M is a compact Kähler manifold then for any integer $1\leq k\leq n$ the closed form ω^k is not exact and that $H^{2n}_{DR}(X)\neq 0$. (Note that ω is a form associated to metric).

Exercise 2.2. If u is an orientation-preserving isometry of T_X, that is $u\in SO(T_X)$, prove that u preserves the Hodge operator. This means the following: u induces an isometry of $T^*_X=\Omega^1_X$ and isometry $\Omega^p u$ of Ω^p_X defined separately on each 1-form. Then for any p-form $\alpha\in\Omega^p$ one has

$$*(\Omega^p u)\alpha=(\Omega^{n-p}u)*\alpha$$

. This illustrates the fact that an orientation-preserving isometry preserves every object canonically attached to a metric and an orientation.

Exercise 2.3. Let E be a holomorphic vector bundle of rank r over a complex manifold X. Show that if L is a holomorphic line bundle on X then $\mathbb{P}(E^*\otimes L^*)=\mathbb{P}(E^*)$ but the line bundle $\mathcal{O}_{\mathbb{P}(E^*\otimes L^*)}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(E^*)}(1)\otimes\pi^*L$ where $\pi:\mathbb{P}(E^*)\to X$ is the structural morphism.