Problem Set 1: Horn's problem and Honeycombs

Anfisa Gurenkova

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> "The only reason for being a bee is to make honey. And the only reason for making honey is so I can eat it."
>
> – Winnie the Pooh

Problem 0. Prove that any Hermitian matrix is diagonalizable and all its eigenvalues are real.

Problem 1. a) Prove the min-max principle: for a Hermitian $n \times n$ matrix A with the spectrum (i.e. the set of eigenvalues) $\lambda_1 \geq \ldots \geq \lambda_n$

$$\lambda_k = \max_{dim L = k} \{ \min_{v \in L \setminus \{0\}} \frac{(Av, v)}{(v, v)} \}$$

$$\lambda_k = \min_{dim L = n-k+1} \{ \max_{v \in L \setminus \{0\}} \frac{(Av, v)}{(v, v)} \}$$

b) Deduce one of Weyl's inequalities: for $A, B, C = A + B$ with spectra $\lambda_i; \mu_i; \nu_i$

$$\nu_k \leq \mu_k + \lambda_1.$$

Problem 2. For a k-dimensional subspace $L \subset \mathbb{C}^n$ we define the *Rayleigh trace* $R_A(L) := \sum_{i=1}^{k} (Av_i, v_i)$ where $\{v_i\}$ is some orthonormal basis of L.

a) Show that it doesn't depend on the choice of the orthonormal basis.

Hint: consider the composite map $L \hookrightarrow \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \xrightarrow{\pi} L$, where π is the orthogonal projection.

b) Show that

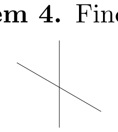
$$\sum_{i=1}^{k} \lambda_i = \max_{dim L = k} R_A(L)$$

c) Deduce Ky Fan's inequalities: for $A, B, C = A + B$ with spectra $\lambda_i; \mu_i; \nu_i$

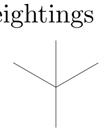
$$\sum_{i=1}^{p} \lambda_i + \sum_{i=1}^{p} \mu_i \geq \sum_{i=1}^{p} \nu_i.$$

Problem 3. Show the trace equality for honeycombs: if λ_i, μ_i, ν_i are the boundary values of the honeycomb, then $\sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i + \sum_{i=1}^{n} \nu_i = 0$

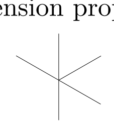
Problem 4. Find all possible weightings satisfying zero-tension property:



a)



b)



b)

Problem 5. Find the conditions for the weights $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2$ to be the boundary values of a 2-honeycomb.

Problem 6. Find all the integral 3-honeycombs with
a) $\lambda = \mu = (1, 0, 0)$; b) $\lambda = (1, 1, 0), \mu = (1, 0, 0)$; c) $\lambda = (2, 0, 0), \mu = (1, 0, 0)$.

Theorem. There exists a triple of Hermitian matrices A, B, C: $A + B + C = 0$ with spectra $\lambda_i, \mu_i, \nu_i \Leftrightarrow$ there exists an n-honeycomb with the boundary values λ_i, μ_i, ν_i.

We write $\lambda \boxplus \mu \boxplus \nu \sim_c 0$ or $\lambda \boxplus \mu \sim_c -\nu$ if the conditions of Theorem are satisfied

Problem 7. Make the first verification of Theorem: if two triples of matrices sum to zero then their direct sum too. Invent a honeycomb analogue of this statement.

Problem 8. Prove the Weyl's inequalities: if $\lambda \boxplus \mu \boxplus \nu \sim_c 0$ then

$$\lambda_i + \mu_j + \nu_k \geq 0$$

for $i + j + k = n + 2$.

Problem 9 (*). Prove the Horn's inequalities: if $\lambda \boxplus \mu \sim_c \nu$ then

$$\sum_I \lambda_i + \sum_J \mu_j \geq \sum_K \nu_k$$

for all subsets I, J, K of $\{1 \dots n\}$ satisfying

- $|I| = |J| = |K| = r < n$ and
- I', J', K' defined by $S' := S - (1, 2 \dots r - 1, r)$ satisfy $I' \boxplus J' \sim_c K'$

Problem 10. Look at the pictures