Farum

P N Kaddaj

February 2023

1 Smooth and Topological Knots

Firstly, why do we care about knots?

Apart from pretty pictures and fun properties they have very important applications to 3 and 4 dimensional topology. For example every 3-manifold is obtained from "Dehn surgery" on a knot in S^3 (here "Dehn surgery" is just an operation of cutting along the knot and gluing something back). In 1966 Fox and Milnor published a paper where they introduced the knot concordance group (via an equivalence relation on knots). To this day still not much is known about this group however understanding of this group is closely related to 4-dimensional topology. An application of this is the distinction between smooth and topological manifolds. i.e. which homeomorphic manifolds admit distinct smooth structures. A structure not equivalent to the standard one is called exotic. Considering a knot with specific properties in the concordance group one can construct an exotic $\mathbb{R}^4$. A currently big open question in the field is whether S^4 admits any exotic structures.

There are many different models for knots adopted according to the context you are working in. So what object should we consider if we want to study knots and what equivalence relation between these objects? Let $Emb(S^1, S^3)$ denote the space of embeddings of S^1 into S^3 and $Smth(S^1, S^3)$ the space of smooth embeddings of S^1 into S^3.

Topological view: Knots are elements of $Emb(S^1, S^3)$

Smooth view: Knots elements of $Smth(S^1, S^3)$

These categories are very different. For example the subspace $Smth(S^1, S^3) \subset Emb(S^1, S^3)$ forms a set of measure zero. i.e. the probability of picking a smooth knot out of topological ones is zero.

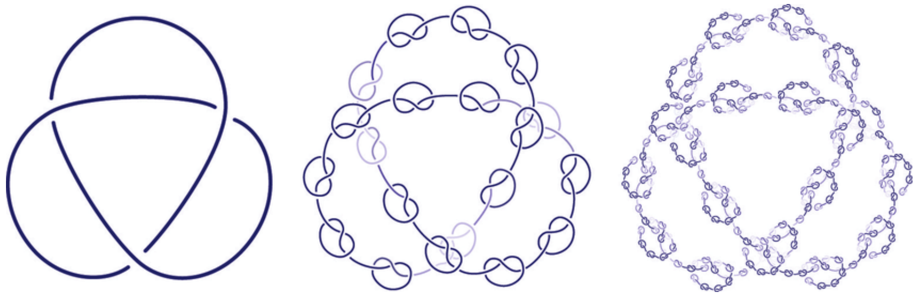

Figure 1: First three iterations of a topological knot

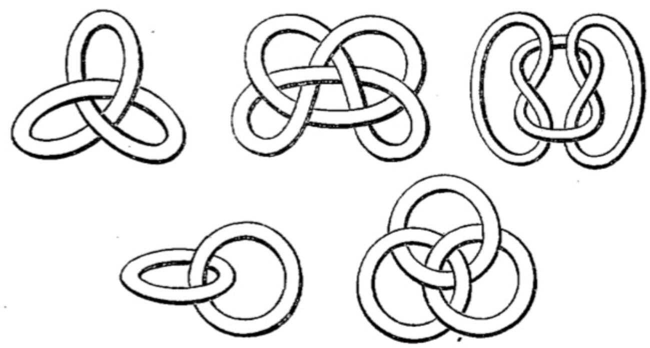

Figure 2: Some smooth knots and links

What equivalence relations should we use in the respective categories? In the smooth view two knots are equivalent if they are connected via a continuous path in $Smth(S^1, S^3)$ (this is called smooth isotopy or sometimes s-isotopy). What about in the topological category? Should we use a similar definition, i.e. a path in $Emb(S^1, S^3)$ (sometimes called topological isotopy or t-isotopy for short?

In fact this may not give what you are looking for aince any smooth knot is t-isotopic to the unknot. Even some very strange topological knots are t-isotopic to the unknot. This leads to a large open problem in knot theory.

Problem 1. *(Rolfsen's Problem) Is the space $Emb(S^1, S^3)$ path connected?*

Let's start with some elementary preliminaries. We often want to consider an algebraic structure on the space of knots, this will allow us to understand properties of knots. One of the simplest structures is connected sum of knots. For this to be well defined we will need a different model for knots i.e. long knots. This is the subspace of $Smth(\mathbb{R}^1, \mathbb{R}^3)$ that have a fixed direction at infinity. Then each such knot can be drawn in a cylinder in a unique way and connected sum is defined by stacking cylinders together. Check that this is well defined and gives a commutative monoid structure on the space of smooth knots (are there potential problems with this definition in the topological category?).

Why doesn't this operation have an inverse? There is a simple argument by Mazur often called the Mazure swindle. However this only works in the topological category.

Exercise 1. *Find an argument that works in the smooth category.*

A more complicated algebraic structure is knot concordance. The idea somewhat orginiates from Artin. In 1926 Artin constructed a knotted embedding of S^2 into $\mathbb{R}^4$ (i.e. the complement of the fundamental group of the image of the embedding is not standard). Intersecting such an embedding with a hyperplane in $\mathbb{R}^4$ gives a knot $K \subset \mathbb{R}^3$. A knot that can be obtained in such a manner is called slice. There arises a natural question as to which knots are slice. For some time it was conjectured that all knots are slice, however in the 1960s Murasaki, Fox and Milnor presented

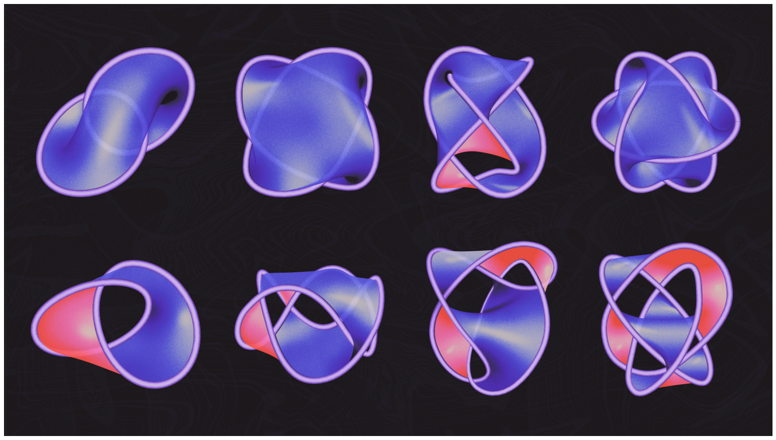

Figure 3: Siefert surfaces for knots and links

example of knots that are not slice. This is another place where the distinction between the smooth and topological categories becomes evident. First we give a formal definition of what it means to be slice.

Definition 1. *(Slice) A knot is slice if it bounds a disk in D^4. Specifically a knot is topologically slice if it the disc it bounds is topological (i.e. not necessarily smooth) and smooth slice if it is smooth.*

Then it is not hard to show that any knot is topologically slice by taking the cone over it. However this construction does not guarantee smoothness since there are often singularities at the vertex of the cone. This example also shows that there is no reasonable extention to 1-dimensional knots in S^4, since in S^3 we define the unkot as the class of knots that bound a disc in S^3 but in S^4 every knot bounds a disc.

There are invariants that help determine whether a knot is slice.

Theorem 1. *(Fox-Milnor Condition) The Alexander polynomial of a slice knot has the form:*

$$A(t) = p(t)p(\frac{1}{t})$$

with $t \in \mathbb{Z}[t]$

So below we introduce the Alexander polynomial. First we start off with the Siefert Pairing.

A general knot $K \subset S^3$ does not bound a disc but bounds a surface called a Siefert surface. This surface is not unique. For example the Mobius strip is a Siefert surface for the trefoil knot. Siefert showed the existence of such a surface by writing down an explicit algorithm.

Exercise 2. *Does there exist a knot K that has a countable number of Siefert surfaces? Is this true for any knot? Try to come up with a relationship between two Siefert surfaces for the same knot K.*

Exercise 3. *Consider the Borromean rings B. Does there exist a disconnected 2-Siefert surface for B. That is can you find surfaces F_1, F_2 s.t. F_i are disjoint in S^3 but bound separate components of B?*

Definition 2. *(Siefert pairing) Let $K \subset F$ be a knot and it's Siefert surface. Then there is a map:*

$$V : H_1(F) \times H_1(F) \to (Z), (x, y) \mapsto lk(x, i_*(y))$$

Where $i : F \to S^3 \setminus F$ is the pushoff map and lk denotes the linking number.

This map is not unique and is not a knot invariant however it can be modified into one. After a choice of basis in $H_1(F)$, V will be a matrix with entries $v_i j$.

Definition 3. *(Alexander polynomial) Given knot $K \subset S^3$ we define the Laurent polynomial:*

$$A_K(t) = det(V^T - tV)$$

Up to multiplication by $\{+/-t^n\}$ for $n \in \mathbb{Z}$

From this definition it is absolutely not clear why this doesn't depend on the choice of surface, bases, etc. However there is another more invariant construction using infinite cyclic covers from which it is more evident that the polynomial is well defined.

Exercise 4. *Let K be a knot in S^3. Let $S_K = S^3 \setminus O(K)$, where $O(K)$ is the interior of a tubular neighbourhood of K. Construct an infinite covering space $S \to S_K$ i.e. a covering space where the fibre of a point is $\mathbb{Z}$.*

We only need one extra ingredient to be able to prove the Fox-Milnor condition. That is that the we can always choose a basis so that V has a triangular block form. From this it a matter of algebra to finish the proof. There is a related statement due to Freedman.

Theorem 2. *(Freedman) A knot K is locally flat topologically slice if it $A_K(t) = 1$*

Another invariant that is associated to a knot is the slice genus i.e. the minimal genus of all Siefert surfaces with boundary K. In practice it is extremely difficult to compute, however we can bound it! An example of this is Rasmussen's s-invariant.

Theorem 3. *(Rasmussen) There is a number s associated to a knot K s.t:*

$$s(K) \leq 2g_*(K)$$

where g_* is the smooth slice genus.

This allows us to give an example of a knot that is topologically slice but not smooth slice. For example it is known that the for the trefoil knot $s = 2$ so $g_* \geq 1$, but the Alexander polynomial of the trefoil knot is equivalent to 1.

To a knot K we can associate a space X_K formed by gluing D^4 to $D^2 \times D^2$ along K. There is a result relating embedding of X_K into $\mathbb{R}^4$ and the slice type of K.

Theorem 4. *There exists smooth (resp. topological) embeddings of $X_K \to \mathbb{R}^4$ iff K is smoothly (resp. locally flat topologically) slice.*

Now pick a knot that is topologically slice but not smoothly slice. Then there is a topological embedding $\phi : X_K \to \mathbb{R}^4$. Next we consider $M_K = \mathbb{R}^4 \setminus \phi(Int(X_K)$, this carries a smooth structure as a 4-manifold. Furthermore ∂M_K and ∂X_K are diffeomorphic since they are homeomorphic as 3-manifolds (3-manifolds admit a unique smooth structure). Thus we can glue X_K back with M_K to make a space R homeomorphic to $\mathbb{R}^4$. If there were a diffeomorphism $R \to \mathbb{R}^4$ it would restrict to a smooth embedding $X_K \to \mathbb{R}^4$ which would contradict the theorem above. Thus we have:

Corollary 1. *(Exotic $\mathbb{R}^4$) There exist exotic structures on $\mathbb{R}^4$*

Exercise 5. *(*) Does there exist a manifold M (of any dimension) that admits no smooth structure?*