Projective Geometry

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1. Affine Transformations

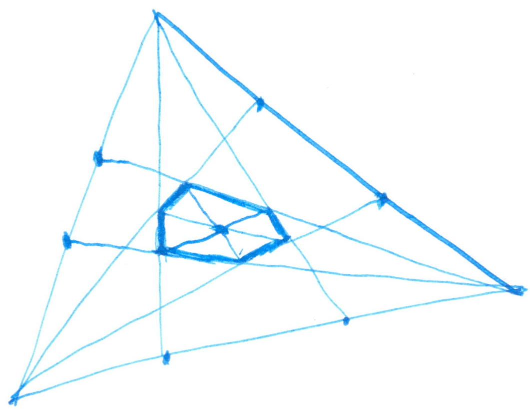
Definition 1.1. An *affine transformation* is a map from the plane to itself that

- is a bijection,
- maps lines to lines (if three points are colinear, they will be mapped to three colinear points),
- is continuous.

1.1. Prove that ratios are preserved by affine transformations.

1.2. Prove that we can map any three points to any other three points with an affine transformation. Prove that an affine transformation is uniquely determined by the image of three points.

1.3. Through every vertex of a triangle two lines are drawn. The lines divide the opposite side of the triangle into three equal parts. Prove that the diagonals connecting opposite vertices of the hexagon formed by these lines intersect at one point.

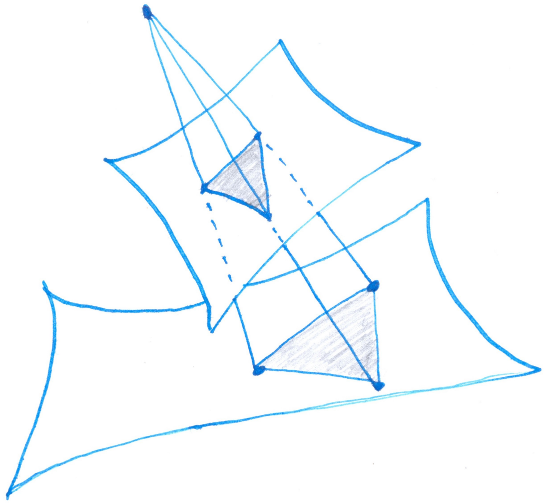


1.4. What conic sections can be mapped to each other by an affine transformation? What about complex affine transformations? What is a complex affine transformation anyways?

- A circle: Ex. $x^2+y^2=1$.
- An ellipse: Ex. $(x-1)^2+2y^2=4$.
- A hyperbola: Ex. $x^2-2y^2=1$.
- A parabola: Ex. $y=x^2+3$.

2. Projective Transformations

Definition 2.1. We can map one plane to another by projection through a point in space not lying on either plane:



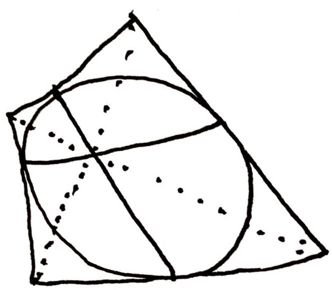
A *projective transformation*, is a combination of projections as shown above, an affine transformations.

2.1. Prove that projective transformations map lines to lines. Is a projective transformation affine?

2.2. When you do a projection, there is a line that disappears. What happens? Try to draw some geometry problems you know, where you let a line disappear. For instance, Desargues' Theorem: Assume that lines AA', BB', CC' intersect. Prove that $AB \cap A'B'$ and $BC' cap B'C'$ and $CA \cap C'A'$ are colinear.

2.3. Prove that, using a projective transformation, you can map any point to the center of a circle. Assume the point lies inside the circle if you do not wish to go complex.

2.4. Solve the problem below. Or even better, remember an old problem (or find one in the pdf *Geometry in Figures*) that is very simple using a projective transformation.



3. Cross Ratios

Definition 3.1. The cross ratio between four points (a, b, c, d), is the ratio $(a-c)/(a-d)$ after d is sent to infinity. A transformation that preserves cross ratios is called *projective.*

3.1. Prove that cross ratios are well defined and preserved under projective transformations. Prove that

$$(a, b, c, d) = (c, d, a, b) = -(b, a, c, d).$$

3.2. Project a line to a conic, through a point on the conic. Project the points from the conic to another line through a different point on the conic. Prove that cross ratios are preserved under this transformation.

3.3. Steiner's Definition of a Conic
Identify the sets of lines P_1, P_2 through two points p_1, p_2, with two projective lines. Consider a projective transformation $\phi : P_1 \to P_2$. Prove that the set of all intersection points $\ell \cap \phi(\ell)$ with $\ell \in \mathbb{P}_1$ form a conic.

3.4. Prove that a conic is uniquely determined by 5 distinct points, with no 3 of the points colinear.

4. Higher Dimensional Projective Geometry

Definition 4.1. Identify the set of 2-dimensional planes in 3-dimensional projective space $\mathbb{P}^3$, containing 3 fixed points, with a projective line. This is called a pencil of planes. Consider three pencil of planes P_1, P_2, P_3 varying projectively, that is we have projective maps $\phi_1 : P_1 \to P_2$ and $\phi_2 : P_2 \to P_3$. The locus of the intersections $\Pi \cap \phi_1(\Pi) \cap \phi_2(\Pi)$ where $\Pi \in P_1$ form a *twisted cubic curve.*

4.1. Prove that a twisted cubic curve intersects all planes in three points (possibly with multiplicity if the plane is tangent to the curve).

4.2. Prove that for any two twisted cubic curves, there exists a projective transformation from projective space to itself, that maps the first curve into the other.

4.3. What curves can you get if you project a twisted cubic curve from projective space to a projective plane? What if the point you project through lies on the twisted cubic curve?

Definition 4.2. Consider two lines ℓ_1, ℓ_2 in $\mathbb{P}^3$, and consider a projective transformation $\phi : \ell_1 \to \ell_2$. For all points $p \in \ell_1$, draw the line through p and $\phi(p)$. This is called a pencil of lines, and their union form a quadric surface.

4.4. Prove that a quadric surface is defined by an equation of degree 2. Prove that all surfaces in $\mathbb{P}^3$ defined by a polynomial of degree 2, is a quadric surface.

4.5. Show that a quadric surface has two pencils of lines. Show that all lines on a quadric surface is contained in one of these two pencils.

4.6. You are given three lines in $\mathbb{P}^3$, all disjoint, and not all contained in a quadric surface. How many lines intersect of three of these lines?

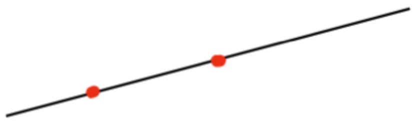
4.7. Identify the set of all lines in $\mathbb{P}^3$ with a quadric surface in $\mathbb{P}^5$.

5. Circular Points at Infinity

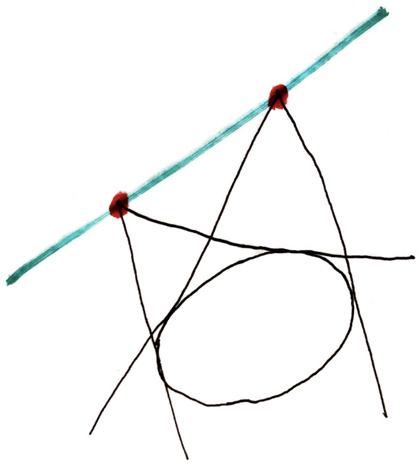
5.1. Prove that there exists 2 complex points at infinity that all circles go through.[1]

5.2. An involution is a projective transformation from a projective line to itself, that is its own inverse [2]. Prove that all involutions preserve two points. Rotating the plane by 90 degrees, induces an involution on the line at infinity. What two points are preserved?

5.3. Draw the radical axis theorem, where the line below is the line at infinity and the two green points are the circular points at infinity.



5.4. Let c_1, c_2 denote the circular points at infinity (in red on the figure below). Draw the tangents from c_1, c_2 to a conic. Prove that the four intersection points of the 4 tangents are the foci points of the conic.



5.5. Consider a pole-polar transformation of a conic in one of its foci. Show that the conic is mapped to a circle.

Theorem 5.1. *Bezout*
Two curves of defined by polynomials of degree d_1 and d_2 has $d_1 \cdot d_2$ intersection points when counting their intersections with multiplicities. For instance, a line intersects a circle in two points, or if it's tangent, in one point with multiplicity two.

5.6. Consider two circles with the same center. Where do they intersect, and how?

[1] Hint: Translate and scale one circle into the other. What happens to the points on the line at infinity?
[2] if you apply the transformation twice, all points end up where they started

6. Linear System of Conics

(Assumes Homogenous Coordinates)
We denote $n-$dimensional projective space by $\mathbb{P}^n$. A conic is given by an equation of the form

$$a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_0x_1+a_4x_1x_2+a_5x_2x_0=0.$$

By identifying a conic with the point $(a_0:a_1:\ldots:a_5)$ in $\mathbb{P}^5$, we can identify the set of conics in $\mathbb{P}^2$ with the set of points in $\mathbb{P}^5$.

6.1. A degenerate conic, is a conic consisting of two lines, or the same line twice. Prove that the set of degenerate conics form a surface Σ in $\mathbb{P}^5$ defined by a polynomial of degree 3.

6.2. Prove that the set of singular points Σ' on Σ can be identified with the conics containing the same line twice. Identify Σ' with $\mathbb{P}^2$.

6.3. Prove that a line in $\mathbb{P}^5$ must either

- intersect Σ in three distinct point not contained in Σ',
- intersect Σ in 1 point with multiplicity two, and another point with multiplicity 1, neither contained in Σ',
- intersect Σ in 1 point with multiplicity 1, not contained in Σ',
- intersect Σ in 1 point on Σ' (with multiplicity 2) and another point not contained in Σ',
- intersect Σ in 1 point on Σ' with multiplicity 3.

In each case, describe the corresponding set of conics in $\mathbb{P}^2$.

6.4. Fix a conic $\mathcal{C}$. Let E denote the surface of conics $\mathcal{C}'$, where $\mathcal{C}$ is Poncelet closed with $\mathcal{C}$ after exactly three steps. Prove that E is obtained by a multiplication in $\mathcal{C}$ of the polar to Σ wrt $\mathcal{C}$, when the 2nd polar to Σ wrt $\mathcal{C}$ is sent to infinity.