Problem Set 3: Schubert calculus on Grassmanians

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Recall that the Grassmanian $Gr(k, V)$ is the set of all k-dimensional subspaces in V. If $V = \mathbb{F}^n$, we write $Gr(k, V) = Gr(k, n)$.

Problem 1. Let $\mathbb{F}_q$ be a finite field of q elements. Find the number of points in $Gr(k, \mathbb{F}_q^n)$.

Problem 2. $GL(V)$ acts on $Gr(k, V)$ in an obvious way. Check that this action is transitive and find the stabiliser of the point $span\{e_1 \dots e_k\}$.

Problem 3. *Fact.* Assume a finite group G acts transitively on a set X. Then $|G| = |X| \cdot |Stab_G(x)|$.

a) $GL_n(\mathbb{F}_q)$ acts transitively on $\mathbb{F}_q^n \setminus \{0\}$. Find the stabilizer of the vector e_1. Then use the Fact to find $|GL_n(\mathbb{F}_q)|$;

b) Now combine the Fact with problem 2 and find $|Gr(k, \mathbb{F}_q^n)|$ again.

Problem 4. Recall the *Plücker embedding*: $Gr(k, V) \to \mathbb{P}(\Lambda^k V)$, $L = span\{v_1 \dots v_k\} \mapsto [v_1 \wedge \dots \wedge v_k]$.

Check that this is indeed an embedding. Find an image of $Gr(2, \mathbb{C}^4)$ in $\mathbb{CP}^5$ under this map (it is given by a system of homogeneous equations).

Problem 5. Find as much ways to find the dimension of $Gr(k, \mathbb{C}^n)$ as you can.

Problem 6. The charts on $Gr(k, \mathbb{R}^n)$ are given by $M_I \neq 0$, where $M_I = M_{i_1 \dots i_k}$ is a minor formed by the columns $i_1 \dots i_k$. Find the transition functions between the charts $M_I \neq 0$ and $M_J \neq 0$ in $Gr(2, \mathbb{R}^4)$.

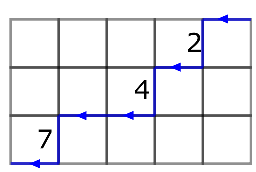
Problem 7 (cell structure on $Gr(k, n)$)**.** Recall the definition of the Schubert cell:

$$\Omega_P(\mathcal{F}_\bullet) = \{L \in Gr(k, n) |\ \dim L \cap \mathcal{F}_q = i \text{ for } p_i \leq q < p_{i+1}\}.$$

a) Show that $\Omega_P(\mathcal{F}_\bullet) \cong \mathbb{F}^N$ and find N.

Hint: you can assume that $\mathcal{F}_\bullet$ is a standard flag. Then it is not hard to write down the general form of the matrix defining L.

Let's assign to $P = p_1 < \dots < p_k$ a Young diagram fitting inside the $k \times n - k$ rectangle: start from the North-East corner and draw a unit interval to the South on each p_i-th step and to the West on all other steps. For example if $k = 3, n = 8, P = (2, 4, 7)$:



b) Show that $\Omega_\lambda \subset \overline{\Omega_\mu}$ iff $\lambda \supset \mu$.

Hint: meditate on the general form of the matrix you obtained in a).

Problem 8 (Schubert cells on $Gr(2, 4)$)**.** Fix a standard flag $\mathcal{F}_\bullet$ in $\mathbb{C}^4$.

a) Interpreting $Gr(2,4)$ as the set of lines in $\mathbb{P}^3$, draw the standard flag in $\mathbb{P}^3$ and for each Schubert cell Ω_I draw the generic point (i.e. line) of it.

b) Draw the graph of incidences between the Schubert cells.

Problem 9. Find all the intersections of the Schubert cells in:
a) $\mathbb{P}^n$, b) $Gr(2,4)$